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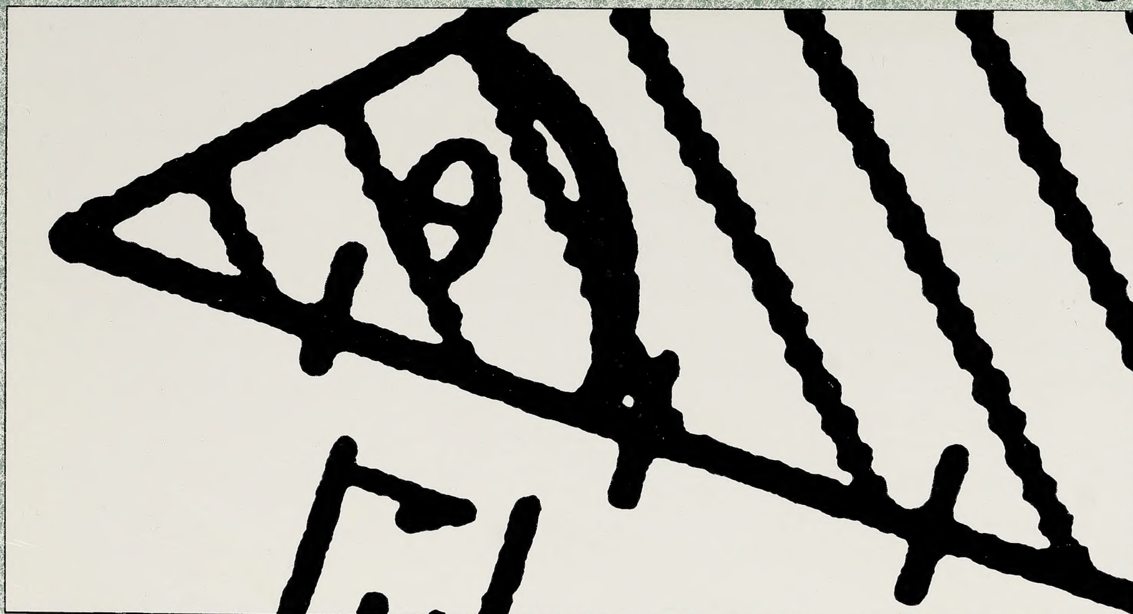
MATHEMATICS 31

Distance Learning



UNIT 7: INNER PRODUCT

Alberta
EDUCATION

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Welcome

Distance Learning

You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.

Mathematics 31 Student Module Unit 7 Inner Product Alberta Distance Learning Centre ISBN No. 0-7741-0294-2

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General Information

This information explains the basic layout of each booklet.

- **What You Already Know** and **Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this unit.

- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.

- **Exploring the Topic** includes the objectives, concept development, and activities for each objective. Use your own papers to arrive at the answers in the activities.

- **Extra Help** reviews the topic. If you had any difficulty with **Exploring the Topic**, you may find this part helpful.

- **Extensions** gives you the opportunity to take the topic one step further.

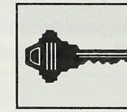
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.

- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics **(Appendix B, etc.)**.

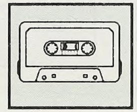
Visual Cues

Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

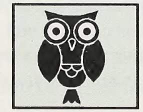
An explanation of what they mean is written beside each visual cue.



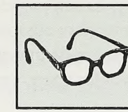
Key Idea
- flagging important ideas



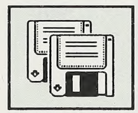
Audiotape
- learning by listening to an audiotape



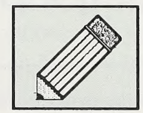
What You Already Know
- reviewing what you already know



Another View
- exploring different perspectives

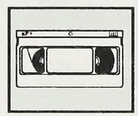


Computer Software
- learning by using computer software



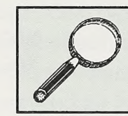
Review
- studying previous concepts

Solutions
- correcting the activities



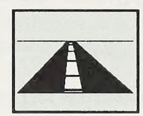
Videotape
- learning by viewing a videotape

Introduction
- introducing the unit

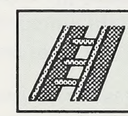


Extra Help
- providing additional study

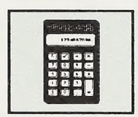
Print Pathway
- choosing a print alternative



What Lies Ahead
- previewing the unit

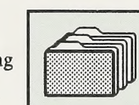


Extensions
- going on with the topic



Calculator
- using your calculator

Exploring the Topic
- actively learning new concepts



What You Have Learned
- summarizing what you have learned

Mathematics 31

Course Overview

Mathematics 31 contains 9 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1 Introduction to Differential Calculus	10%
Unit 2 Differentiation of Algebraic Expression and Graphing	10%
Unit 3 Practical Application of Derivatives	20%
Unit 4 Integration	10%
Unit 5 Geometric Vectors and Their Application	10%
Unit 6 Algebraic Vectors and Their Application	10%
Unit 7 Inner Product	10%
Unit 8 Systems of Linear Equations	10%
Unit 9 Matrices and Linear Transformations	10%
	100%

Unit Assessment

After completing the unit you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal, your teacher will determine what this assessment will be. It may be

Unit Assignment - 50%
Supervised Unit Test - 50%

Introduction to Inner Product

This unit covers topics dealing with inner product. Each topic contains explanations, examples, and activities to assist you in understanding inner product. If you find you are having difficulty with the explanations and the way the material is presented, there is a section called **Extra Help**. If you would like to extend your knowledge of the topic, there is a section called **Extensions**.

You can evaluate your understanding of each topic by working through the activities. Answers are found in the solutions in the **Appendix**. In several cases there is more than one way to do the question.

Unit 7 Inner Product

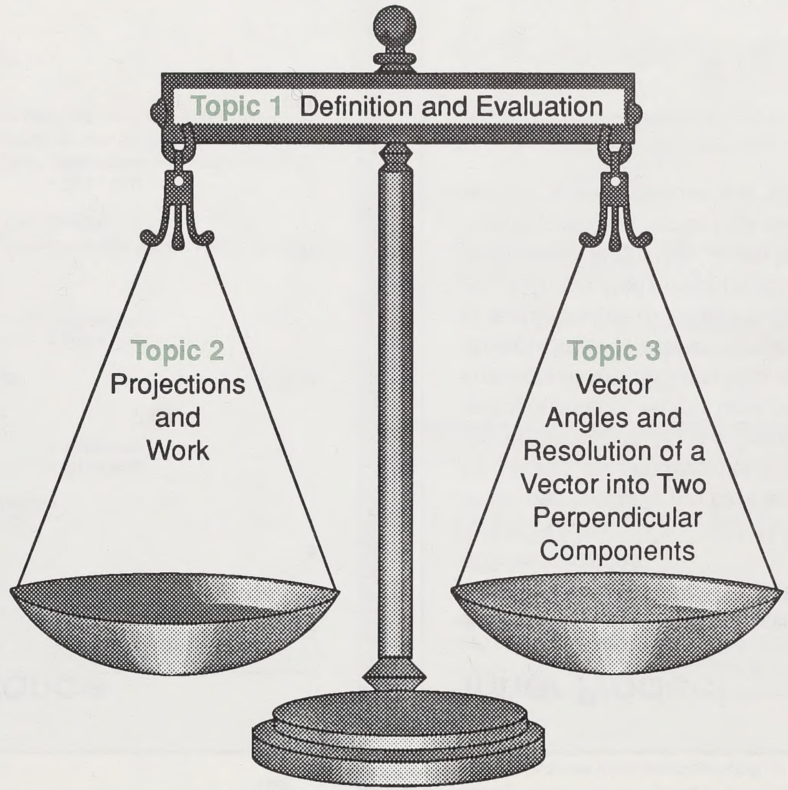
Contents at a Glance

Inner Product

Addition and subtraction of vectors are not like addition and subtraction of numbers. Multiplication of vectors also has a different meaning.

In the previous unit you have learned how to multiply a vector by a scalar, but you have not learned how to multiply a vector by another vector. Two different ways of defining multiplication of vectors have been invented, and each of them corresponds to a different idea in physics. One type of multiplication of vectors is called **inner product**; the other type of multiplication of vectors is called **cross product**. In this unit you will only study inner product. Inner product has many applications in physics. It can be used to find angles between vector quantities, vector projections, perpendicular components of vectors, and the amount of work done in moving an object.

Unit 7 Inner Product



What You Already Know

A good understanding of the concepts from **Unit 5** and **Unit 6** is required to solve many of the problems in this unit. If you are familiar with the following concepts, then proceed with this unit.

- The following two methods are used to find the resultant of adding two vectors which are not collinear.

  – the triangle method
  – the parallelogram method

- The negative or inverse of $\vec{u}$ is a vector equal in magnitude but opposite in direction to $\vec{u}$, and it is denoted by $-\vec{u}$. Therefore, you can define subtraction of vectors as follows:

$$\vec{u}-\vec{v}=\vec{u}+\left(-\vec{v}\right)$$

- The resultant of two or more forces acting at a point on a body is the sum of the given vectors. The magnitude of the resultant can be determined by using the law of cosines.

- The vector $\overrightarrow{PQ}$ determined by $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $[x_2 - x_1, y_2 - y_1]$.

- The magnitude $\left|\vec{v}\right|$ of $\vec{v}=[a,b,c,]$ is $\left|\vec{v}\right|=\sqrt{a^2+b^2+c^2}$.

- A single force may be resolved into components.

- Rectangular components are those components which are perpendicular to each other.

- Addition and subtraction of vectors in two-space is as follows:

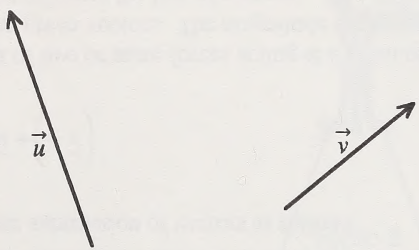
$$[a_1, a_2] \pm [b_1, b_2] = [a_1 \pm b_1, a_2 \pm b_2]$$

Now that you have looked at material that you studied previously, go to the **Review** to confirm your understanding of this material.

Review

1. Refer to the following diagram where $\vec{u}$ and $\vec{v}$ are two vectors.

   Find $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ geometrically.



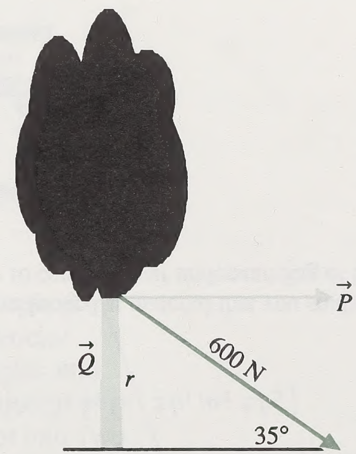
2. Three forces of 20 N, 30 N, and 40 N act on a body. The first two act at an angle of 60° to each other, and the third force is perpendicular to the first two vectors. Find the magnitude of the resultant to the nearest newton.

3. Find the vector $\overrightarrow{PQ}$ given $P(1,-3,5)$ and $Q(4,0,-2)$.

4. Find the length of the vector $\vec{u}=[3,-2,1]$.

5. Multiply vector $[3,-2,1]$ by 3.

6. A tree is anchored by means of a guy wire as shown. If the tension in the guy wire is 600 N, what are the horizontal $|\vec{P}|$ and vertical $|\vec{Q}|$ components?



7. Find $\vec{A}+\vec{B}$ if $\vec{A}=[3,1]$ and $\vec{B}=[2,5]$.

Now go to the **Review** solutions in the **Appendix**.

Topic 1 Definition and Evaluation

Introduction

Multiplication of numbers is derived from addition. Since $3+3=6$, then $2\times3=6$. If you want to multiply a vector by another vector, this concept **cannot** be used. A vector is not a number. A vector has magnitude and direction.

A vector can be in two-space or three-space. Two types of multiplication of vectors have been invented. One type is called **inner product**. In this topic you will learn the definition of inner product of algebraic vectors and learn to simplify expressions using inner product.

What Lies Ahead

Throughout the topic you will learn to

1. define inner product in three different ways, and calculate expressions using inner product

Now that you know what to expect, turn the page to begin your study of definition and evaluation.

Exploring Topic 1

Activity 1

Define inner product in three different ways, and calculate expressions using inner product.

What is inner product? Inner product is sometimes called **dot product** because the symbol used to denote this operation is a dot.

(For example, $\vec{A} \bullet \vec{B}$ is used to denote the inner product of vectors $\vec{A}$ and $\vec{B}$.)

It is also called **scalar product** because the product is a scalar.

In order to give some concrete interpretation to inner product, it will be defined in three ways. It will be defined in algebraic terms, in geometric terms, and in descriptive terms. The algebraic definition leads to the geometric definition, and the geometric definition leads to the descriptive definition.

Inner Product Defined Algebraically

In algebraic terms the inner product for the algebraic vectors $\vec{A} = [a_1, a_2]$ and $\vec{B} = [b_1, b_2]$ (two-dimensional vectors) is defined as follows:

$$\vec{A} \bullet \vec{B} = a_1 b_1 + a_2 b_2$$

Similarly, the inner product for three-dimensional vectors is as follows:

If $\vec{A} = [a_1, a_2, a_3]$ and $\vec{B} = [b_1, b_2, b_3]$, then

$\vec{A} \bullet \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Note that $a_1 b_1 + a_2 b_2$ and $a_1 b_1 + a_2 b_2 + a_3 b_3$ are scalars. According to this definition the dot product of two vectors is a scalar quantity and not a vector.

The dot product is not closed under the multiplication. By this statement it means that when you multiply two vectors together, the product is not a vector.

Now look at some examples.

Example 1

Find $\vec{A} \cdot \vec{B}$ if $\vec{A} = [3, -5]$ and $\vec{B} = [1, 4]$.

Solution:

$$\begin{aligned} \vec{A} \cdot \vec{B} &= [3, -5] \cdot [1, 4] \\ &= 3(1) + (-5)(4) \\ &= 3 - 20 \\ &= -17 \end{aligned}$$

Example 2

Find $\vec{A} \cdot \vec{B}$ if $\vec{A} = [2, 4, -3]$ and $\vec{B} = [-2, 0, 5]$.

Solution:

$$\begin{aligned} \vec{A} \cdot \vec{B} &= [2, 4, -3] \cdot [-2, 0, 5] \\ &= (2)(-2) + (4)(0) + (-3)(5) \\ &= -4 + 0 - 15 \\ &= -19 \end{aligned}$$

Example 3

Find $\vec{A} \cdot \vec{A}$ if $\vec{A} = [1, 2, 4]$.

Solution:

$$\begin{aligned} \vec{A} \cdot \vec{A} &= [1, 2, 4] \cdot [1, 2, 4] \\ &= (1)^2 + (2)^2 + (4)^2 \\ &= 1 + 4 + 16 \\ &= 21 \end{aligned}$$

Example 4

Find m so that $\vec{A} \cdot \vec{B} = 4$ if $\vec{A} = [5, m, 1]$ and $\vec{B} = [2, -1, -2]$.

Solution:

$$\begin{aligned} \vec{A} \cdot \vec{B} &= 4 \\ [5, m, 1] \cdot [2, -1, -2] &= 4 \\ (5)(2) + m(-1) + (1)(-2) &= 4 \\ 10 + (-m) - 2 &= 4 \\ 8 - m &= 4 \\ m &= 4 \end{aligned}$$

Note that $\vec{A} \cdot \vec{A} = \left|\vec{A}\right|^2$.

$$\begin{aligned} \left|\vec{A}\right| &= \sqrt{1^2 + 2^2 + 4^2} \\ &= \sqrt{21} \\ \therefore \left|\vec{A}\right|^2 &= \vec{A} \cdot \vec{A} = 21 \end{aligned}$$

Example 5

Show that the inner product is commutative for $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$ if $\vec{A} = [1, 2, -2]$ and $\vec{B} = [3, 0, 1]$.

Solution:

$$\begin{aligned} \vec{A} \bullet \vec{B} &= [1, 2, -2] \bullet [3, 0, 1] \\ &= 1(3) + 2(0) + (-2)(1) \\ &= 3 + 0 - 2 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \vec{B} \bullet \vec{A} &= [3, 0, 1] \bullet [1, 2, -2] \\ &= (3)(1) + 0(2) + (1)(-2) \\ &= 3 + 0 - 2 \\ &= 1 \end{aligned}$$

Therefore, $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$.

Note that Example 5 is not a proof. It represents only one special case.

Example 6

Show that the inner product is distributive for $\vec{A} \bullet \left(\vec{B} + \vec{C}\right) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$ if $\vec{A} = [1, 2, -2]$, $\vec{B} = [3, 0, 1]$, and $\vec{C} = [4, -1, 5]$.

Solution:

$$\begin{aligned} \vec{A} \bullet \left(\vec{B} + \vec{C}\right) &= [1, 2, -2] \bullet \{[3, 0, 1] + [4, -1, 5]\} \\ &= [1, 2, -2][7, -1, 6] \\ &= (1)(7) + (2)(-1) + (-2)(6) \\ &= 7 - 2 - 12 \\ &= -7 \end{aligned}$$

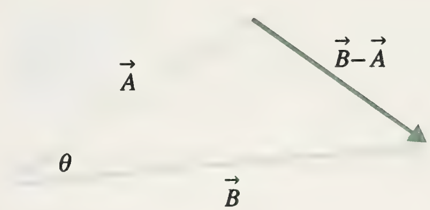
$$\begin{aligned} \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C} &= [1, 2, -2] \bullet [3, 0, 1] + [1, 2, -2] \bullet [4, -1, 5] \\ &= 1(3) + 2(0) + (-2)(1) + (1)(4) + (2)(-1) + (-2)(5) \\ &= 3 + 0 - 2 + 4 - 2 - 10 \\ &= 7 - 14 \\ &= -7 \end{aligned}$$

Therefore, $\vec{A} \bullet \left(\vec{B} + \vec{C}\right) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$.

Inner Product Defined Geometrically

In geometric terms the algebraic definition of dot product leads to the geometric definition. This is done by utilizing the law of cosines for triangles.

If $\vec{A}=[a_1, a_2]$, $\vec{B}=[b_1, b_2]$, and θ is the angle between the two vectors, then the lengths of the sides are $|\vec{A}|$, $|\vec{B}|$, and $|\vec{B}-\vec{A}|$.



The following can be stated according to the law of cosines.

$$|\vec{B}-\vec{A}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta$$

$$\therefore 2|\vec{A}||\vec{B}|\cos\theta = |\vec{A}|^2 + |\vec{B}|^2 - |\vec{B}-\vec{A}|^2$$

Since $\vec{A}=[a_1, a_2]$ and $\vec{B}=[b_1, b_2]$, you can solve for $\vec{B}-\vec{A}$.

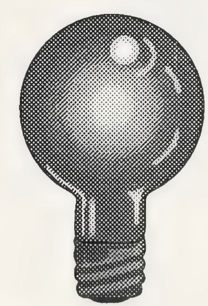
$\vec{B}-\vec{A}=[b_1-a_1, b_2-a_2]$ (vector subtraction)

$|\vec{A}|^2 = a_1^2 + a_2^2$ (Recall that $|\vec{A}|=\sqrt{a_1^2+a_2^2}$ where $\vec{A}=[a_1, a_2]$.)

$|\vec{B}|^2 = b_1^2 + b_1^2$

Therefore, $|\vec{A}|^2 = a_1^2 + a_2^2$, $|\vec{B}|^2 = b_1^2 + b_2^2$,

and $|\vec{B}-\vec{A}|^2 = (b_1-a_1)^2 + (b_2-a_2)^2$.



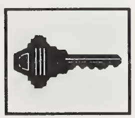
As a result you get the following:

$$
\begin{aligned}
2|\vec{A}||\vec{B}|\cos\,\theta &= |\vec{A}|^2 + |\vec{B}|^2 - |\vec{B}-\vec{A}|^2 \\
&= a_1{}^2 + a_2{}^2 + b_1{}^2 + b_2{}^2 - (b_1 - a_1)^2 - (b_2 - a_2)^2 \\
&= a_1{}^2 + a_2{}^2 + b_1{}^2 + b_2{}^2 - b_1{}^2 + 2b_1a_1 - a_1{}^2 - b_2{}^2 + 2b_2a_2 - a_2{}^2 \\
&= 2a_1b_1 + 2a_2b_2 \\
&= 2(a_1b_1 + a_2b_2)
\end{aligned}
$$

$$\therefore |\vec{A}||\vec{B}|\cos\,\theta = a_1b_1 + a_2b_2 \quad \left(\text{This is the algebraic definition of } \vec{A}\bullet\vec{B}.\right)$$

$$= \vec{A}\bullet\vec{B}$$

Now you can define the dot product in geometric terms.



$\vec{A}\bullet\vec{B} = |\vec{A}||\vec{B}|\cos\,\theta$, where θ is the angle between the vectors.

This definition allows for the possibility of a negative dot product since $\cos\,\theta$ can be negative if θ is not acute.

How do you use this definition in geometric terms?

Look at some more examples.

Example 7

Determine θ $\left(\text{the angle between } \vec{A} \text{ and } \vec{B}\right)$ to the nearest degree, and verify that $\vec{A} \bullet \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \cos\ \theta$ where $\vec{A} = [-3, 0]$ and $\vec{B} = [0, 5]$.

Solution:

Since $\vec{A}$ is on the x-axis and $\vec{B}$ is on the y-axis, θ is 90°.

$$\left|\vec{A}\right| = \sqrt{(-3)^2 + 0^2} \qquad \left|\vec{B}\right| = \sqrt{0^2 + 5^2}$$
$$= 3 \qquad\qquad = 5$$

$$\begin{aligned}\left|\vec{A}\right|\left|\vec{B}\right| \cos\ \theta &= \left|\vec{A}\right|\left|\vec{B}\right| \cos\ 90° \\ &= 3(5)(0) \\ &= 0\end{aligned}$$

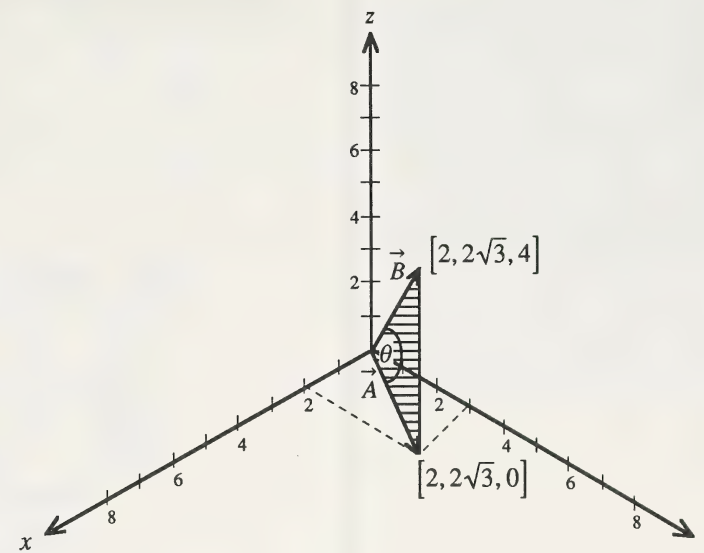
$$\begin{aligned}\vec{A} \bullet \vec{B} &= [-3, 0] \bullet [0, 5] \quad \text{(algebraically)} \\ &= (-3)0 + 0(5) \\ &= 0\end{aligned}$$

Therefore $\left|\vec{A}\right|\left|\vec{B}\right| \cos\ \theta = \vec{A} \bullet \vec{B}$ where $\theta = 90°$.

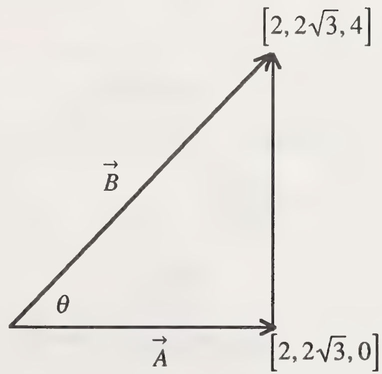
Example 8

Determine θ $\left(\text{the angle between } \vec{A} \text{ and } \vec{B}\right)$ to the nearest degree, and verify that $\vec{A} \bullet \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \cos\ \theta$ where $\vec{A} = \left[2, 2\sqrt{3}, 0\right]$ and $\vec{B} = \left[2, 2\sqrt{3}, 4\right]$.

Solution:



The x- and y-coordinates of $\vec{A}$ and $\vec{B}$ are equal, and the z-coordinate of $\vec{A}$ is 0. $\vec{A}$ is in the xy-plane. The vectors $\vec{A}$ and $\vec{B}$ determine a right triangle.



The previous diagram is a side view of $\vec{A}$ and $\vec{B}$.

$$\left|\vec{A}\right| = \sqrt{4+12+0}$$
$$= \sqrt{16}$$
$$= 4$$

$$\left|\vec{B}\right| = \sqrt{4+12+16}$$
$$= \sqrt{32}$$
$$= 4\sqrt{2}$$

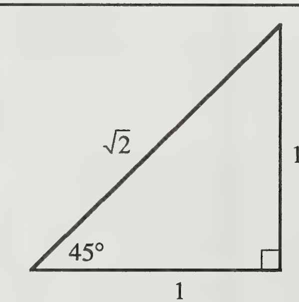
$$\cos\ \theta = \frac{\left|\vec{A}\right|}{\left|\vec{B}\right|}$$
$$= \frac{4}{4\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}$$
$$\theta = 45°$$

$$\therefore \left|\vec{A}\right|\left|\vec{B}\right| \cos\ \theta = 4\left(4\sqrt{2}\right)(\cos 45°)$$
$$= 4\left(4\sqrt{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$
$$= 16$$

$$\vec{A} \bullet \vec{B} = \left[2, 2\sqrt{3}, 0\right] \bullet \left[2, 2\sqrt{3}, 4\right] \text{ (algebraically)}$$
$$= 4 + 12 + 0$$
$$= 16$$

Therefore, $\vec{A} \bullet \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \cos\ \theta$ where $\theta = 45°$.

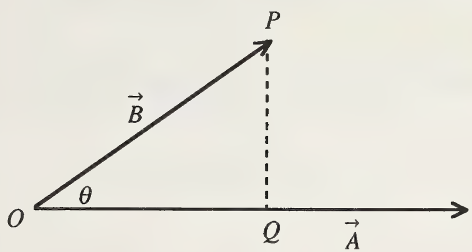
The definition of dot product was invented so that it could be used to solve problems in physics. What does the definition $\vec{A} \bullet \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \cos\ \theta$ represent? Look at the descriptive definition.



$$\cos\ 45° = \frac{1}{\sqrt{2}}$$

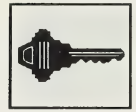
Inner Product Defined Descriptively

In descriptive terms inner product is the magnitude of the projection of one vector on another vector multiplied by the magnitude of the second vector. (This definition only holds for vectors at an acute angle with each other.)



According to the geometric definition of dot product, $\vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}| \cos\ \theta$.

In the previous diagram $\vec{B} \cos\ \theta = |\overrightarrow{OQ}|$, which is the magnitude of the projection of vector $\vec{B}$ on $\vec{A}$. Now you can see that the inner product of two vectors $\vec{A}$ and $\vec{B}$ is the magnitude of the projection of $\vec{B}$ on $\vec{A}$ multiplied by the magnitude of $\vec{A}$.



If $\vec{B}$ represents a force and $\vec{A}$ represents displacement, then $\vec{A} \bullet \vec{B}$ represents the component of a force $\left(\vec{B}\right)$ in the direction of $\vec{A}$ times a distance $|\vec{A}|$.

In physics this product represents **work**. Since this part involves a lot more explanations and study, you will learn more about work in the next topic. However, if you are curious and want to know more about cross product, go the **Extensions** section.

$$\cos\ \theta = \frac{|\overrightarrow{OQ}|}{|\vec{B}|}$$

Do the following exercise. Do either the odd-numbered or even-numbered questions.

1. Calculate $\vec{A} \cdot \vec{B}$ for the following pairs of vectors.

   a. $\vec{A} = [-3, 1]$ and $\vec{B} = [5, 2]$

   b. $\vec{A} = [2, -1, 5]$ and $\vec{B} = [0, 3, 6]$

   c. $\vec{A} = [a, \pi, 1]$ and $\vec{B} = [b, \pi, 2]$

2. Calculate $\vec{F} \cdot \vec{S}$ for the following pairs of vectors.

   a. $\vec{F} = [5, 1]$ and $\vec{S} = [-2, 3]$

   b. $\vec{F} = [3, 1, 2]$ and $\vec{S} = [-2, 0, 1]$

   c. $\vec{F} = [a, 2, \pi]$ and $\vec{S} = \left[b, 3, \frac{1}{\pi}\right]$

3. a. Find $\vec{A} \cdot \vec{A}$ if $\vec{A} = [3, -2]$.

   b. Find $\vec{A} \cdot \vec{A}$ if $\vec{A} = [0, -1, 5]$.

4. a. Find $\vec{B} \cdot \vec{B}$ if $\vec{B} = [-3, -2]$.

   b. Find $\vec{B} \cdot \vec{B}$ if $\vec{B} = [-2, -1, 3]$.

5. Find m so that $[3, -5, 2m] \cdot [3, 2, -1] = -2$.

6. Find n so that $[n, -2, 1] \cdot [3, 1, n] = 4$.

7. Calculate the following given $\vec{A} = [3, -1]$, $\vec{B} = [5, 2]$, and $\vec{C} = [0, 4]$.

   a. $3\vec{A} \cdot \vec{B}$

   b. $\left(\vec{A} - \vec{B}\right) \cdot \vec{C}$

8. Calculate the following given $\vec{D} = [5, -1]$, $\vec{E} = [1, 3]$, and $\vec{F} = [2, 2]$.

   a. $\vec{D} \cdot \vec{E} + \vec{F} \cdot \vec{D}$

   b. $3\vec{D} \cdot \left(\vec{E} + \vec{F}\right)$

9. Verify the following given $\vec{A}=[3,1,2]$ and $\vec{B}=[-1,-2,2]$.

a. $3\vec{B}\bullet\vec{A}=\vec{B}\bullet\left(3\vec{A}\right)$

b. $\vec{A}\bullet\vec{A}=\left|\vec{A}\right|^2$

10. Verify the following given $\vec{C}=[5,1,0]$ and $\vec{D}=[-2,1,2]$.

a. $\vec{C}\bullet\left(5\vec{D}\right)=\left(5\vec{C}\right)\bullet\vec{D}$

b. $\vec{D}\bullet\vec{D}=\left|\vec{D}\right|^2$

11. Determine θ (the angle between the two given vectors) by observation. Then, verify $\vec{A}\bullet\vec{B}=\left|\vec{A}\right|\left|\vec{B}\right|\cos\,\theta$ for the following pairs of vectors. (See Example 7 and Example 8.)

a. $\vec{A}=[-3,0]$ and $\vec{B}=[0,5]$

b. $\vec{A}=[-3,6]$ and $\vec{B}=[-9,18]$

c. $\vec{A}=[2,2,0]$ and $\vec{B}=\left[2,2,2\sqrt{2}\right]$

12. Determine θ (the angle between the two given vectors) by observation. Then verify $\vec{A}\bullet\vec{B}=\left|\vec{A}\right|\left|\vec{B}\right|\cos\,\theta$.

a. $\vec{A}=[-5,0]$ and $\vec{B}=[0,4]$

b. $\vec{A}=[-2,4]$ and $\vec{B}=[-6,12]$

c. $\vec{A}=\left[6,8,10\sqrt{3}\right]$ and $\vec{B}=[6,8,0]$

13. Find the angle θ between $\vec{A}$ and $\vec{B}$ given $\vec{A}=[3,1,-2]$ and $\vec{B}=[5,0,7]$.

14. Find the angle θ between $\vec{u}$ and $\vec{v}$ given $\vec{u}=[-3,-5,1]$ and $\vec{v}=[0,-2,3]$.

For solutions to **Activity 1**, turn to the **Appendix, Topic 1**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

You may decide to do both.

Extra Help

Inner product is also called dot product or scalar product. $\vec{A}\bullet\vec{B}$ is used to denote the inner product of vectors $\vec{A}$ and $\vec{B}$. Inner product can be explained by the following three definitions:

- algebraic definition

  For algebraic vectors $\vec{A}=[a_1, a_2]$ and $\vec{B}=[b_1, b_2]$ it can be stated that $\vec{A}\bullet\vec{B}=a_1b_1+a_2b_2$.

  If $\vec{A}=[1,2]$ and $\vec{B}=[3,4]$, then the following can be stated:

$$\begin{aligned}\vec{A}\bullet\vec{B}&=(1\times3)+(2\times4)\\&=3+8\\&=11\end{aligned}$$

  If $\vec{A}=[1,3,5]$ and $\vec{B}=[2,4,6]$, then the following can be stated:

$$\begin{aligned}\vec{A}\bullet\vec{B}&=(1\times2)+(3\times4)+(5\times6)\\&=2+12+30\\&=44\end{aligned}$$

- geometric definition

  The algebraic definition of inner product leads to the geometric definition. (See the proof in Activity 1.)

  $\vec{A}\bullet\vec{B}=\left|\vec{A}\right|\left|\vec{B}\right|\cos\,\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$.

  If $\vec{A}=[0,5]$, which is on the y-axis, and $\vec{B}=[3,0]$, which is on the x-axis, then θ is a 90° angle (x-axis and y-axis are perpendicular).

$$\begin{aligned}\vec{A}\bullet\vec{B}&=\left|\vec{A}\right|\left|\vec{B}\right|\cos\,\theta\\&=\left|\sqrt{0+5^2}\right|\left|\sqrt{3^2+0}\right|\cos\,90^\circ\\&=5(3)(0)\\&=0\end{aligned}$$

- descriptive definition

The geometric definition shows that $|\vec{A}||\vec{B}|\cos\,\theta$ represents the magnitude of the component of $\vec{B}$ in the direction of $\vec{A}$ multiplied by the magnitude of $\vec{A}$. This product actually represents the work done by a force. Work is an application of inner product and will be discussed in **Topic 2**.

Now try the following questions.

1. Calculate $\vec{A} \bullet \vec{B}$ if $\vec{A} = [2, 3]$ and $\vec{B} = [1, 6]$.

2. Find $\vec{A} \bullet \vec{A}$ if $\vec{A} = [2, 3]$.

3. Calculate $\vec{A} \bullet \vec{B}$ if $\vec{A} = [0, 2, 1]$ and $\vec{B} = [3, 2, 7]$.

4. Given $\vec{A} = [-3, 0]$ and $\vec{B} = [0, -2]$, show that $\vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}|\cos\,\theta$.

   (You have to determine θ by observation.)

5. Find the angle θ between $\vec{A}$ and $\vec{B}$ if $\vec{A} = [5, -1]$ and $\vec{B} = [3, 2]$.

For solutions to **Extra Help**, turn to the **Appendix**, **Topic 1**.

Extensions

The other type of vector multiplication is called **cross product** (or vector product). Cross product will be defined only in three-space.

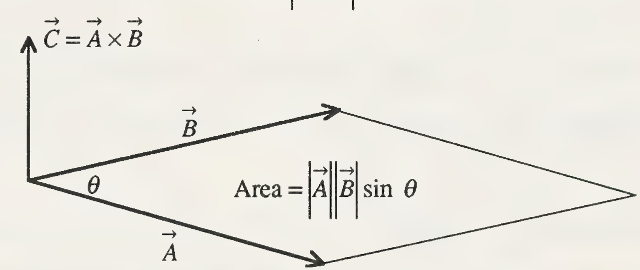
If $\vec{A}$ and $\vec{B}$ are two vectors, the cross product of these two vectors is denoted by $\vec{A} \times \vec{B}$. This product is a **vector**. (Inner product is a scalar.)

The magnitude of a cross product is defined as follows:

$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin\,\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$.

The vector $\vec{A} \times \vec{B}$ is perpendicular to the plane of $\vec{A}$ and $\vec{B}$.

The area of the parallelogram $|\vec{A} \times \vec{B}|$ is determined by $\vec{A}$ and $\vec{B}$.



In algebraic terms the cross product of $\vec{A}=[a_1, a_2, a_3]$ and $\vec{B}=[b_1, b_2, b_3]$ is this:

$$\vec{A}\times\vec{B}=[a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1]$$

This definition is derived by the following method.

Suppose $\vec{C}=[c_1, c_2, c_3]$ is perpendicular to $\vec{A}$ and $\vec{B}$. The dot products are $\vec{A}\bullet\vec{C}=0$ and $\vec{B}\bullet\vec{C}=0$.

$$\vec{A}\bullet\vec{C}=a_1c_1+a_2c_2+a_3c_3=0 \quad (1)$$
$$\vec{B}\bullet\vec{C}=b_1c_1+b_2c_2+b_3c_3=0 \quad (2)$$

$$a_1\times(2): a_1b_1c_1+a_1b_2c_2+a_1b_3c_3=0 \quad (3)$$
$$b_1\times(1): a_1b_1c_1+a_2b_1c_2+a_3b_1c_3=0 \quad (4)$$

$$(3)-(4): a_1b_2c_2-a_2b_1c_2+a_1b_3c_3-a_3b_1c_3=0$$
$$a_1b_2c_2-a_2b_1c_2=a_3b_1c_3-a_1b_3c_3$$
$$c_2(a_1b_2-a_2b_1)=c_3(a_3b_1-a_1b_3)$$
$$\frac{c_2}{a_3b_1-a_1b_3}=\frac{c_3}{a_1b_2-a_2b_1}$$

By a similar method you can prove that $\frac{c_1}{a_2b_3-a_3b_2}=\frac{c_2}{a_3b_1-a_1b_3}$.

Now you can state the following:

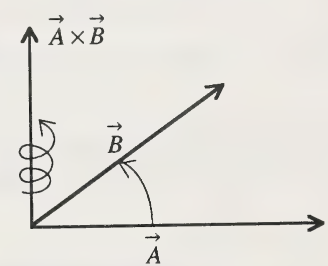
$$\frac{c_1}{a_2b_3-a_3b_2}=\frac{c_2}{a_3b_1-a_1b_3}=\frac{c_3}{a_1b_2-a_2b_1}=k \quad (k \text{ is a constant.})$$

When $k=1$, $\vec{A}\times\vec{B}$ is the vector. If k is any real number other than 1, then k represents a vector collinear with $\vec{C}$ but with a different magnitude.

$$c_1=a_2b_3-a_3b_2$$
$$c_2=a_3b_1-a_1b_3$$
$$c_3=a_1b_2-a_2b_1$$

$$\therefore \vec{C}=[a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1]$$

$\vec{A}\times\vec{B}$ is perpendicular to $\vec{A}$ and $\vec{B}$ such that $\vec{A}$, $\vec{B}$, and $\vec{A}\times\vec{B}$ form a **right-hand system.** $\vec{A}\times\vec{B}$ is the normal (perpendicular line) obtained by the motion of a right-hand screw when $\vec{A}$ is rotated into $\vec{B}$.



Cross product can be used in physics. Look at the following examples.

Example 9

Find the area of the parallelogram determined by $\vec{A} = [3, 2, 5]$ and $\vec{B} = [0, -1, 4]$.

Solution:

$$\vec{A} \times \vec{B} = [a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1]$$

$$\vec{A} \times \vec{B} = [(2)(4)-(5)(-1),\ (5)(0)-(3)(4),\ (3)(-1)-(2)(0)]$$
$$= [13, -12, -3]$$

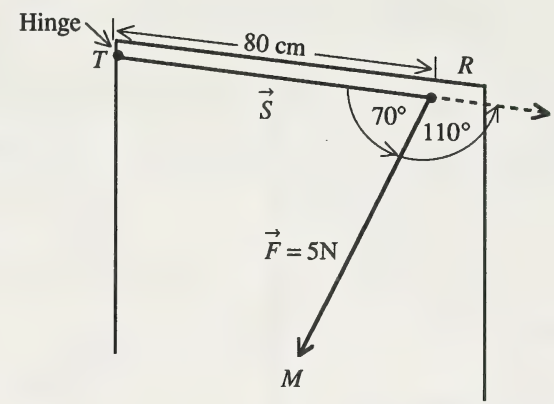
$$\left|\vec{A} \times \vec{B}\right| = \sqrt{13^2 + (-12)^2 + (-3)^2}$$
$$= \sqrt{169 + 144 + 9}$$
$$= \sqrt{322}$$
$$\doteq 17.94$$

The area is approximately 17.94 units^2.

Example 10

A 5 N force is applied at a point 80 cm from a hinge of a door. The door and the force form a 70° angle. Find the magnitude of the moment (torque) about the hinge.

Solution:



In the diagram let $\vec{S} = \overrightarrow{TR}$ and $\vec{F} = \overrightarrow{RM}$.

$$\left|\vec{S}\right| = 80 \text{ cm}$$
$$= 0.80 \text{ m}$$
$$\left|\vec{F}\right| = 5 \text{ N}$$

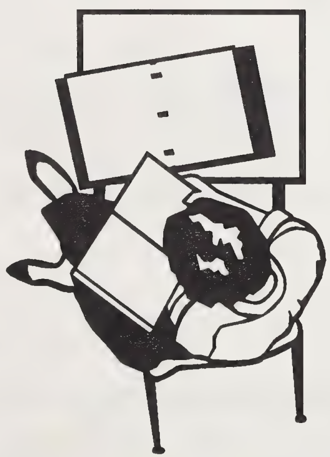
The angle between vector $\vec{S}$ and vector $\vec{F}$ is 110°.

Let $\theta = 110°$.

The vector $\vec{S} \times \vec{F}$ is the moment of the force (torque).

The magnitude of the moment is $3.76 \text{ N}\bullet\text{m}$.

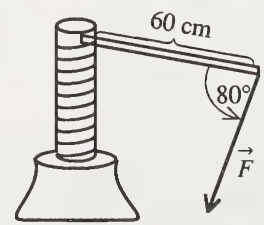
$$\begin{aligned} \left|\vec{S} \times \vec{F}\right| &= \left|\vec{S}\right|\left|\vec{F}\right| \sin\theta \\ &= (0.80)(5)\sin 110° \\ &\doteq (4)(0.9397) \\ &\doteq 3.76 \end{aligned}$$



Now try the following questions.

1. Find the vector which is perpendicular to $\vec{A} = [3, 5, -1]$ and $\vec{B} = [-2, -1, -1]$.

2. A force of 30 N is applied to the end of a handle of a jack-screw. The jack handle is 60 cm long. The force and the handle form an 80° angle. Calculate the magnitude of the moment about the other end of the handle.



3. Find the area of the parallelogram determined by $\vec{A} = [3, 5, -2]$ and $\vec{B} = [-3, 0, 9]$.

For solutions to **Extensions**, turn to the **Appendix, Topic 1**.

The product of the magnitude of the force and the perpendicular distance from the centre of rotation to the force is called the moment of the force.

The perpendicular distance between the hinge and the force is $\left|\vec{S}\right| \sin\theta$.

Topic 2 Projections and Work

Introduction

The projection of one vector on another vector is the **component** of the first vector. If this component represents a component of a force and the second vector represents the displacement of an object, then the inner product represents the **work** done by this force. This is what you are going to learn about in this topic.

What Lies Ahead

Throughout the topic you will learn to

1. define scalar projection of a vector and solve related problems

2. define work and solve related problems

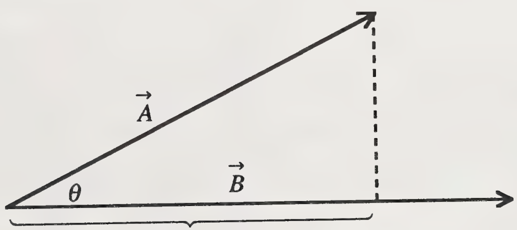
Now that you know what to expect, turn the page to begin your study of projections and work.

Exploring Topic 2

Activity 1

Define scalar projection of a vector and solve related problems.

If $\vec{A}$ and $\vec{B}$ are two vectors and θ is the angle between $\vec{A}$ and $\vec{B}$, then the **scalar projection** of $\vec{A}$ on $\vec{B}$ is the **scalar** that cuts $\vec{B}$ with a perpendicular from $\vec{B}$ to $\vec{A}$.



Scalar projection of $\vec{A}$ on $\vec{B} = |\vec{A}| \cos\theta$

Since $\cos\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}}$, the projection (adjacent side) is $|\vec{A}| \cos\theta$, where $|\vec{A}|$ is the length of vector $\vec{A}$.

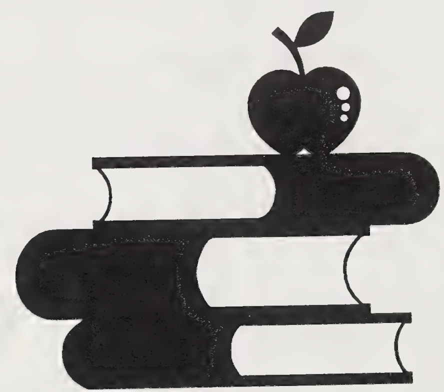
The scalar projection of vector $\vec{A}$ on vector $\vec{B}$ is a scalar.

According to the definition of dot product, $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$.

Therefore, $|\vec{A}| \cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|\vec{B}|}$.

The scalar projection of $\vec{A}$ on $\vec{B}$ is $\dfrac{\vec{A} \cdot \vec{B}}{|\vec{B}|}$.

It is often called a **component**.



Now look at some examples.

Example 1

Given $\vec{A}=[3,6,5]$ and $\vec{B}=[0,1,2]$, find the scalar projection of $\vec{B}$ on $\vec{A}$.

Solution:

The scalar projection of $\vec{B}$ on $\vec{A}$ is as follows:

$$\begin{aligned}\frac{\vec{A}\bullet\vec{B}}{|\vec{A}|} &= \frac{[3,6,5]\bullet[0,1,2]}{\sqrt{3^2+6^2+5^2}}\\ &= \frac{0+6+10}{\sqrt{9+36+25}}\\ &\doteq \frac{16}{8.3666}\\ &\doteq 1.912\end{aligned}$$

The dot product of the two vectors can be positive or negative. A negative projection occurs when θ is obtuse (greater than 90°).

Example 2

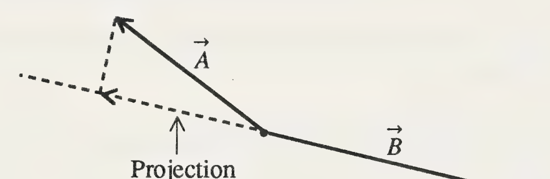
Find the projection of $\vec{A}=[-2,5]$ on $\vec{B}=[3,-1]$.

Solution:

The scalar projection of $\vec{A}$ on $\vec{B}$ is as follows:

$$\begin{aligned}\frac{\vec{A}\bullet\vec{B}}{|\vec{B}|} &= \frac{[-2,5]\bullet[3,-1]}{\sqrt{3^2+(-1)^2}}\\ &= \frac{-6-5}{\sqrt{10}}\\ &= -\frac{11}{\sqrt{10}}\\ &\doteq -3.48\end{aligned}$$

The fact that the projection is negative suggests that the direction of the projection is opposite to that of $\vec{B}$.



Therefore, the scalar projection of $\vec{A}$ on $\vec{B} = \frac{\vec{A}\bullet\vec{B}}{|\vec{B}|} = \frac{-11}{\sqrt{10}} \doteq -3.48$.

Now do the odd- or even-numbered questions.

1. Find the scalar projection of $\vec{A}$ on $\vec{B}$.

   a. $\vec{A}=[3,1]$ $\quad \vec{B}=[0,-2]$

   b. $\vec{A}=[2,-1,3]$ $\quad \vec{B}=[5,0,-4]$

2. Find the scalar projection of $\vec{A}$ on $\vec{B}$.

   a. $\vec{A}=[-2,-3]$ $\quad \vec{B}=[-5,8]$

   b. $\vec{A}=[5,6,-2]$ $\quad \vec{B}=[3,0,7]$

3. The vertices of ΔPQR are $P(3,5)$, $Q(-2,-4)$, and $R(-2,3)$.

   Show that the sum of the scalar projection of $\overrightarrow{PR}$ on $\overrightarrow{PQ}$ and the projection of $\overrightarrow{RQ}$ on $\overrightarrow{PQ}$ is equal to the length of $\overrightarrow{PQ}$.

4. The vertices of ΔPQR are $P(-2,2)$, $Q(3,-2)$, and $R(-4,-1)$.

   Show that the sum of the scalar projection of $\overrightarrow{PQ}$ on $\overrightarrow{PR}$ and the projection of $\overrightarrow{QR}$ on $\overrightarrow{PR}$ is equal to the length of $\overrightarrow{PR}$.

5. What condition must exist so that the projection of $\vec{A}$ on $\vec{B}$ is zero?

6. What condition must exist so that the projection of $\vec{A}$ on $\vec{B}$ equals the projection of $\vec{B}$ on $\vec{A}$?

For solutions to **Activity 1**, turn to the **Appendix, Topic 2**.

Activity 2

Define work and solve related problems.

One of the most important applications of vectors in physics is work.

Work is defined as the product of a force applied to an object multiplied by the distance which it moves. In other words **work = applied force × distance**.

The force and distance must be in the **same direction**. If the force is in newtons (N) and the distance in metres (m), the product is a derived unit called the newton metre $(\text{N}\bullet\text{m})$. A newton metre is also called a **joule** (J).

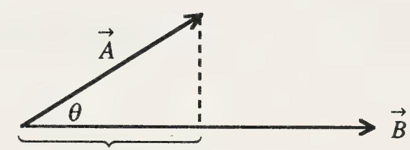
One joule of work is done when a force of 1 N acts on an object through a distance of 1 m.

If a horizontal force of 5 N is used to push a box 2.0 m along a level floor, then the amount of work done is $5\text{ N} \times 2.0\text{ m} = 10\text{ J}$.

The dot product of two vectors $\vec{A}$ and $\vec{B}$ at an angle θ to each other is $\vec{A}\bullet\vec{B} = |\vec{A}||\vec{B}|\cos\ \theta$.

Now, if $\vec{A}$ represents a force acting at an angle θ in the direction an object is moved by the force, and if $\vec{B}$ represents the distance the object is moved, then $|\vec{A}|\cos\ \theta$ represents the projection of $\vec{A}$ on $\vec{B}$. This gives the component of the force acting in the direction of motion. Then, $|\vec{A}||\vec{B}|\cos\ \theta$ is the product of the magnitude of the force component and the distance moved. Therefore, it represents the work done.

$$\text{Work} = \vec{A}\bullet\vec{B} = |\vec{A}||\vec{B}|\cos\ \theta$$



Scalar projection of $\vec{A}$ on $\vec{B} = |\vec{A}|\cos\ \theta$

The product $\vec{A}\bullet\vec{B}$ is a positive or negative scalar depending on the sign of $\cos\theta$. If the product is negative, it indicates that the force $\vec{A}$ has been applied at an obtuse angle with the vector $\vec{B}$; thus, the object would move in the direction opposite to the direction of vector $\vec{B}$. You should note that although force and distance are vector quantities, work is a scalar quantity.

If $\vec{A}=[a_1, a_2]$ and $\vec{B}=[b_1, b_2]$, then

Work $=\vec{A}\bullet\vec{B}=a_1b_1+a_2b_2$.

If $\vec{A}=[a_1, a_2, a_3]$ and $\vec{B}=[b_1, b_2, b_3]$, then

Work $=a_1b_1+a_2b_2+a_3b_3$.

Now look at some examples.

Example 3

A force (in newtons) given by $\vec{F}=[3,5]$ moves an object (in metres) from the point $P(-2,1)$ to the point $Q(4,6)$. Find the work done.

Solution:

$$\begin{aligned}\overrightarrow{PQ}&=[4-(-2), 6-1]\\&=[6,5]\end{aligned}$$

$$\begin{aligned}\text{Work}&=\vec{F}\bullet\overrightarrow{PQ}\\&=[3,5]\bullet[6,5]\\&=(3)(6)+(5)(5)\\&=18+25\\&=43\end{aligned}$$

The work done is 43 J.

In the next example the geometric definition of dot product is used to determine the work done.

Example 4

Find the work done by a 20 N force if it moves an object (in metres) from $P(2,3,1)$ to $Q(-5,0,4)$. The force is acting at an angle of 60° to $\overrightarrow{PQ}$.

Solution:

$$\begin{aligned}\overrightarrow{PQ}&=[-5-2, 0-3, 4-1]\\&=[-7,-3,3]\end{aligned}$$

$$\begin{aligned}\left|\overrightarrow{PQ}\right|&=\sqrt{(-7)^2+(-3)^2+(3)^2}\\&=\sqrt{67}\end{aligned}$$

$$\begin{aligned}\text{Work}&=(20)\left|\overrightarrow{PQ}\right|\cos 60^\circ\\&=20\left(\sqrt{67}\right)(0.5)\\&=10\sqrt{67}\end{aligned}$$

The word done is $10\sqrt{67}$ J.

Example 5

Find the work done by a 50 N force if it moves an object (in metres) from $P(-2,-3,-4)$ to $Q(3,2,5)$. The force is acting in the direction of $\overrightarrow{MN}$ with $M(0,0,3)$ and $N(0,0,8)$.

Solution:

$$\begin{aligned}\overrightarrow{PQ} &= [3+2, 2+3, 5+4] \\ &= [5,5,9]\end{aligned}$$

$$\begin{aligned}\overrightarrow{MN} &= [0-0, 0-0, 8-3] \\ &= [0,0,5]\end{aligned}$$

Now find $\vec{F}$ describing the force. The force is parallel to $\overrightarrow{MN}$.

$$\begin{aligned}|50| &= k\sqrt{0+0+25} \\ k &= \frac{50}{5} \\ &= 10\end{aligned}$$

$$\begin{aligned}\vec{F} &= k[0,0,5] \\ &= 10[0,0,5] \\ &= [0,0,50]\end{aligned}$$

$$\begin{aligned}\text{Work} &= \vec{F}\bullet\overrightarrow{PQ} \\ &= [0,0,50]\bullet[5,5,9] \\ &= 0+0+450 \\ &= 450\end{aligned}$$

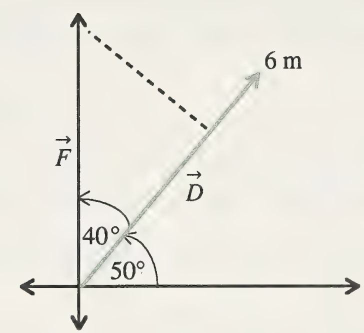
The work done is 450 J.

Try a more difficult example.

Example 6

Determine the work done by a 150 N person climbing a staircase which is 6 m long and forms an angle of 50° with the horizontal.

Solution:



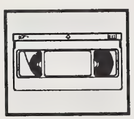
(Use $\cos\ 40°$ because it is the angle between the two vectors.)

Standing still, the person is exerting a downward force $-\vec{F}$ of 150 N. A force greater than $+\vec{F}$ must be exerted in order to move vertically upward. A force greater than the component of $+\vec{F}$ must be exerted in order to move in the direction of the staircase.

The actual work done is slightly greater than the following:

$$\begin{aligned}\vec{F}\bullet\vec{D} &= \left|\vec{F}\right|\left|\vec{D}\right|\cos 40^\circ \\ &\doteq 150(6)(0.766) \\ &\doteq 689.4\end{aligned}$$

The work done is approximately 689.4 J.



If you have access to a videocassette recorder (VCR), you may wish to view the video titled **Dot Product and Projections** for a reinforcement of **Topic 1** and **Topic 2**. This video is program 15 in the *Catch 31*[1] series.

Now do the odd- or even-numbered questions.

1. A force (in newtons) is given by $\vec{F} = [4, -3]$. The force moves an object along the entire distance (in metres) of the vector $\vec{S} = [2, 7]$. Calculate the work done.

[1] *Catch 31* is a title of ACCESS Network.

2. A force (in newtons) is given by $\vec{F} = [-2, 5]$. The force moves an object along the entire distance (in metres) of the vector $\vec{S} = [-3, 4]$. Calculate the work done.

3. Find the work done by a 30 N force if it moves an object (in metres) from $P(5, 3, 1)$ to $Q(9, 0, 8)$. The force is acting in the direction of $\overrightarrow{MN}$ with $M(2, 1, 4)$ and $N(0, 3, 1)$.

4. Find the work done by a 40 N force if it moves an object (in metres) from $P(2, -2, 1)$ to $Q(3, 5, 7)$. The force is acting at an angle of 45° to $\overrightarrow{PQ}$.

5. A sled is pulled 30 m along level ground by a rope which makes an angle of 30° with the horizontal. If it is pulled with a force of 25 N, find the work done.

6. Determine the amount of work done by an 80 N child climbing a staircase which is 5 m long and forms an angle of 60° with the horizontal.

For solutions to **Activity 2**, turn to the **Appendix, Topic 2**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

The projection of a vector is a component of a vector. If $\vec{A}$ and $\vec{B}$ are two vectors, then the projection of $\vec{A}$ on $\vec{B}$ is equal to $\frac{\vec{A} \bullet \vec{B}}{|\vec{B}|}$, and the projection of $\vec{B}$ on $\vec{A}$ is equal to $\frac{\vec{A} \bullet \vec{B}}{|\vec{A}|}$ where $\vec{A} \bullet \vec{B}$ is the dot product of $\vec{A}$ and $\vec{B}$. The magnitudes of vector $\vec{A}$ and vector $\vec{B}$ are $|\vec{A}|$ and $|\vec{B}|$.

Example 7

Find the scalar projection of $\vec{B}$ on $\vec{A}$ given $\vec{A} = [2, 5]$ and $\vec{B} = [1, 1]$.

Solution:

$$\frac{\vec{A} \bullet \vec{B}}{|\vec{A}|} = \frac{[2, 5] \bullet [1, 1]}{\sqrt{2^2 + 5^2}}$$

$$= \frac{2+5}{\sqrt{29}}$$

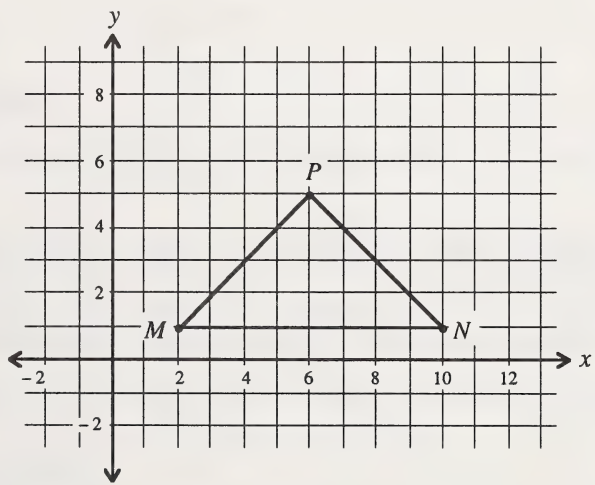
$$= \frac{7}{\sqrt{29}}$$

The projection of $\vec{B}$ on $\vec{A}$ is $\frac{7}{\sqrt{29}}$.

Example 8

Figure MNP is an isosceles triangle. The vertices are $M(2,1)$, $N(10,1)$, and $P(6,5)$. Show that the projection of $\overrightarrow{MP}$ on $\overrightarrow{MN}$ is equal to the projection of $\overrightarrow{PN}$ on $\overrightarrow{MN}$.

Solution:



$$\overrightarrow{MP} = [6-2,\ 5-1] \\ = [4,4]$$

$$\overrightarrow{PN} = [10-6, 1-5] \\ = [4,-4]$$

$$\overrightarrow{MN} = [10-2, 1-1] \\ = [8,0]$$

Find the projection of $\overrightarrow{MP}$ on $\overrightarrow{MN}$.

$$\frac{\overrightarrow{MP} \bullet \overrightarrow{MN}}{\left|\overrightarrow{MN}\right|} = \frac{[4,4] \bullet [8,0]}{\sqrt{8^2+0}} \\ = \frac{32+0}{8} \\ = 4$$

Find the projection of $\overrightarrow{PN}$ on $\overrightarrow{MN}$.

$$\frac{\overrightarrow{PN} \bullet \overrightarrow{MN}}{\left|\overrightarrow{MN}\right|} = \frac{[4,-4] \bullet [8,0]}{\sqrt{8^2+0}} \\ = \frac{32+0}{8} \\ = 4$$

Therefore, the projection of $\overrightarrow{MP}$ on $\overrightarrow{MN}$ is equal to the projection of $\overrightarrow{PN}$ on $\overrightarrow{MN}$.

Work is the dot product of two vectors. It is also the scalar projection of one vector on another multiplied by the magnitude of the second vector.

If $\vec{A}$ represents a force and $\vec{B}$ represents a distance vector, then

work $= \vec{A} \bullet \vec{B}$.

If $\vec{A} = [a_1, a_2]$, $\vec{B} = [b_1, b_2]$, and θ is the angle between $\vec{A}$ and $\vec{B}$,

then work $= \vec{A} \bullet \vec{B} = (a_1 b_1 + a_2 b_2)$, or work $= |\vec{A}||\vec{B}| \cos \theta$.

Example 9

A force $\vec{F}$ is expressed by $\vec{F} = [2, 6]$. A directed distance (in metres) is expressed by $\vec{S} = [-2, 7]$. If the force $\vec{F}$ (in newtons) moves an object in the direction of $\vec{S}$, find the work done.

Solution:

$$\begin{aligned} \text{Work} &= [2,6] \bullet [-2,7] \\ &= 2(-2) + 6(7) \\ &= -4 + 42 \\ &= 38 \end{aligned}$$

The work done is 38 J.

Example 10

A child pulls a sled with a rope a distance of 10 m along level ground. Find the work done if the tension in the rope is 25 N and the rope makes an angle of 60° with the horizontal.

Solution:

$|\vec{F}| = 25$ N

$|\vec{S}| = 10$ m

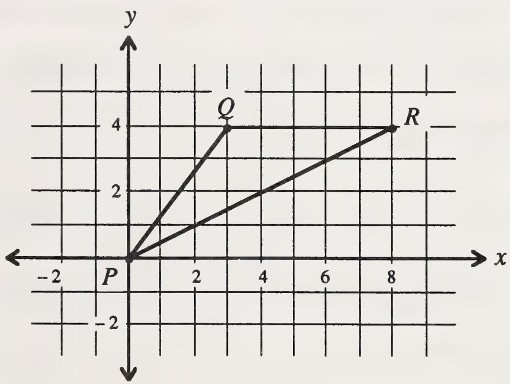
$$\begin{aligned} \text{Work} &= |\vec{F}||\vec{S}| \cos 60^\circ \\ &= (25)(10) \cos 60^\circ \\ &= 125 \end{aligned}$$

The work done is 125 J.

Try the following questions.

1. Find the scalar projection of $\vec{B}$ on $\vec{A}$ given $\vec{A}=[7,8]$ and $\vec{B}=[-2,-5]$.

2. Figure PQR is an isosceles triangle. The vertices are $P(0,0)$, $Q(3,4)$, and $R(8,4)$. Show that the projection of $\overrightarrow{PQ}$ on $\overrightarrow{PR}$ is equal to the projection of $\overrightarrow{QR}$ on $\overrightarrow{PR}$.



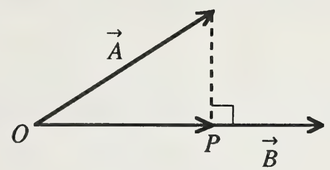
3. A force is expressed by $\vec{F}=[-3,5]$. A directed distance (in metres) is expressed by $\vec{S}=[-2,9]$. The force $\vec{F}$ (in newtons) moves an object in the direction of $\vec{S}$. Find the work done.

4. A 25 N force moves an object (in metres) from $A(3,3)$ to $B(-2,6)$. If the angle between the force and $\overrightarrow{AB}$ is 70°, find the work done.

For solutions to **Extra Help**, turn to the **Appendix**, **Topic 2**.

Extensions

In this topic you have learned the scalar projection of a vector.



In the previous diagram if $\vec{A}$ is projected on $\vec{B}$, then the magnitude of $\overrightarrow{OP}$ is called the scalar projection of $\vec{A}$. The vector $\overrightarrow{OP}$ is called the **vector projection** of $\vec{A}$ on $\vec{B}$ and is the scalar projection $\left(\text{magnitude of } \overrightarrow{OP}\right)$ multiplied by a unit vector in the direction of $\vec{B}$.

$$\begin{aligned}\overrightarrow{OP} &= \text{vector projection of } \vec{A} \text{ on } \vec{B}\\ &= \left(\frac{\vec{A}\cdot\vec{B}}{\left|\vec{B}\right|}\right)\left(\frac{\vec{B}}{\left|\vec{B}\right|}\right)\\ &= \frac{\vec{A}\cdot\vec{B}}{\left|\vec{B}\right|^2}\left(\vec{B}\right)\end{aligned}$$

The unit vector in the direction of $\vec{B}$ is equal to $\frac{\vec{B}}{\left|\vec{B}\right|}$.

Example 11

Find the vector projection of $\vec{A}$ on $\vec{B}$ given $\vec{A}=[3,7]$ and $\vec{B}=[-5,2]$.

Solution:

$$\frac{\vec{A}\bullet\vec{B}}{|\vec{B}|^2}\left(\vec{B}\right)=\left[\frac{[3,7]\bullet[-5,2]}{\left|\sqrt{(-5)^2+2^2}\right|^2}\right][-5,2]$$

$$=\frac{(3)(-5)+(7)(2)}{29}[-5,2]$$

$$=\frac{-1}{29}[-5,2]$$

$$=\left[\frac{5}{29},-\frac{2}{29}\right]$$

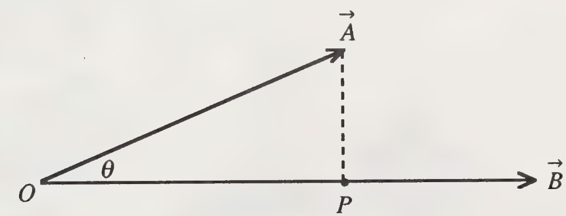
The vector projection of $\vec{A}$ on $\vec{B}$ is $\left[\frac{5}{29},-\frac{2}{29}\right]$.

Try the following questions.

1. Find the vector projection of $\vec{B}$ on $\vec{A}$ given $\vec{A}=[5,0,-3]$ and $\vec{B}=[1,8,3]$.

2. Find the vector projection of $\vec{A}=[-3,5,1]$ on the y-axis.

3. The vector $\overrightarrow{OP}$ is the vector projection of $\vec{A}$ on $\vec{B}$. Express the vector projection of $-3\vec{A}$ on $2\vec{B}$ in terms of $\overrightarrow{OP}$.



For solutions to **Extensions**, turn to the **Appendix, Topic 2**.

Topic 3 Vector Angles and Resolution of a Vector into Two Perpendicular Components

Introduction

In this topic you will solve problems involving angles determined by two vectors in two- or three-space. A vector is like a force. In physics a force is often resolved into components which are the effective values of the force in specific directions. You are also going to learn vector resolution in this topic.

What Lies Ahead

Throughout the topic you will learn to

1. determine vector angles

2. resolve a vector into two perpendicular components

Now that you know what to expect, turn the page to begin your study of vector angles and resolution of a vector into two perpendicular components.

Exploring Topic 3

Activity 1

Determine vector angles.

Two vectors are equivalent as long as they have the same magnitude and direction; therefore, any vector can be represented geometrically by an equivalent vector with its initial point at the origin. If two vectors are given, they can be represented by two vectors with their initial points at the origin. The angle between the two original vectors is represented by the angle between the two equivalent vectors with initial points at the origin.

If two vectors are given in algebraic form, the definition of dot product provides us with the means to determine the angle between the two vectors.

The dot product of $\vec{A}$ and $\vec{B}$ is as follows:

$$\vec{A}\bullet\vec{B}=\left|\vec{A}\right|\left|\vec{B}\right|\cos\,\theta\;\left(\theta \text{ is the angle between } \vec{A} \text{ and } \vec{B}\right)$$

You may write $\cos\,\theta=\dfrac{\vec{A}\bullet\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}$.

The previous formula can be used to find the angle between $\vec{A}$ and $\vec{B}$.

Example 1

Find the angle θ between $\vec{A}$ and $\vec{B}$ to the nearest degree given $\vec{A}=[3,-10]$ and $\vec{B}=[-5,-6]$.

Solution:

$$\begin{aligned}\vec{A}\bullet\vec{B}&=[3,-10]\bullet[-5,-6]\\&=(3)(-5)+(-10)(-6)\\&=-15+60\\&=45\end{aligned}$$

$$\begin{aligned}\left|\vec{A}\right|&=\sqrt{3^2+(-10)^2}\\&=\sqrt{109}\end{aligned}\qquad\begin{aligned}\left|\vec{B}\right|&=\sqrt{(-5)^2+(-6)^2}\\&=\sqrt{25+36}\\&=\sqrt{61}\end{aligned}$$

$$\begin{aligned}\cos\,\theta&=\frac{45}{\sqrt{109}\sqrt{61}}\\&\doteq 0.5519\\\therefore\theta&\doteq 57^\circ\end{aligned}$$

Therefore, θ is approximately 57°.

The next example uses three-space vectors.

Example 2

Find the angle θ between $\vec{A}$ and $\vec{B}$ given $\vec{A}=[3,10,1]$ and $\vec{B}=[0,2,-5]$.

Solution:

$$\begin{aligned}\vec{A}\bullet\vec{B}&=[3,10,1]\bullet[0,2,-5]\\&=0+20-5\\&=15\end{aligned}$$

$$\begin{aligned}\left|\vec{A}\right|&=\sqrt{3^2+10^2+1^2}\\&=\sqrt{110}\end{aligned} \qquad \begin{aligned}\left|\vec{B}\right|&=\sqrt{0+2^2+(-5)^2}\\&=\sqrt{29}\end{aligned}$$

$$\begin{aligned}\cos\theta&=\frac{\vec{A}\bullet\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}\\&=\frac{15}{\sqrt{110}\sqrt{29}}\\&=\frac{15}{\sqrt{3190}}\\&\doteq 0.2656\\ \therefore\theta&\doteq 74.6^\circ\end{aligned}$$

θ is approximately 74.6°.

To find the angle between two line segments, change the line segments to vectors by finding the difference of the respective coordinates of their end points and then applying the formula

$$\cos\theta=\frac{\vec{A}\bullet\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}.$$

Look at the next example.

Example 3

Segment PQ is determined by $P(5,0,-3)$ and $Q(1,4,-5)$.

Determine the angle made by $\overrightarrow{PQ}$ and the positive y-axis.

Solution:

Let $\vec{A}=\overrightarrow{PQ}$.

$$\begin{aligned}\vec{A}&=[1-5,4-0,-5+3]\\&=[-4,4,-2]\end{aligned}$$

Let $\vec{B}=[0,1,0]$ or any vector with the x- and z-coordinates equal to zero and the y- coordinate positive to represent a vector along the positive y-axis.

$$\begin{aligned}\vec{A}\bullet\vec{B}&=[-4,4,-2]\bullet[0,1,0]\\&=0+4+0=4\end{aligned}$$

$$\left|\vec{A}\right| = \sqrt{(-4)^2 + 4^2 + (-2)^2}$$
$$= \sqrt{16+16+4}$$
$$= \sqrt{36}$$
$$= 6$$

$$\left|\vec{B}\right| = \sqrt{0+1^2+0}$$
$$= 1$$

$$\cos\ \theta = \frac{\vec{A} \bullet \vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}$$
$$= \frac{4}{(6)(1)}$$
$$= 0.\dot{6}$$
$$\therefore \theta \doteq 48.2°$$

The angle is approximatly 48.2°.

Two nonzero vectors $\vec{A}$ and $\vec{B}$ are perpendicular or **orthogonal** if the angle between them is 90°.

Since $\cos\ \theta = \frac{\vec{A} \bullet \vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}$ and $\cos\ 90° = 0$, then

$$\cos\ 90° = \frac{\vec{A} \bullet \vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|} = 0.$$

Two nonzero vectors are orthogonal only if $\vec{A} \bullet \vec{B} = 0$.

Now look at the next example.

Example 4

Find k if $\vec{A} = [4, 5, -3]$ and $\vec{B} = [1, 1, k]$ are orthogonal.

Solution:

If $\vec{A}$ and $\vec{B}$ are orthogonal, then $\vec{A} \bullet \vec{B} = 0$.

$$[4, 5, -3] \bullet [1, 1, k] = 0$$
$$4 + 5 - 3k = 0$$
$$9 = 3k$$
$$k = 3$$

Orthogonal is the word applied to geometric objects (other than lines) that are perpendicular.

Example 5

The vertices of a triangle in three-space are $P(3,-1,2)$, $Q(4,3,-1)$, and $R(3,1,-4)$. Is ΔPQR a right triangle?

Solution:

$$\begin{aligned}\overrightarrow{QP} &= [3-4,-1-3,2-(-1)] \\ &= [-1,-4,3]\end{aligned}$$

$$\begin{aligned}\overrightarrow{QR} &= [3-4,1-3,-4-(-1)] \\ &= [-1,-2,-3]\end{aligned}$$

$$\begin{aligned}\overrightarrow{QP}\cdot\overrightarrow{QR} &= [-1,-4,3]\cdot[-1,-2,-3] \\ &= 1+8-9 \\ &= 0\end{aligned}$$

Thus, $\overrightarrow{QP}$ and $\overrightarrow{QR}$ are orthogonal and $\angle Q = 90°$.
Therefore, ΔPQR is a right triangle.

Now do the odd- or even-numbered questions.

1. Find the angle between $\vec{A}$ and $\vec{B}$ to the nearest degree given vectors $\vec{A}$ and $\vec{B}$.

   a. $\vec{A}=[3,-9]$ and $\vec{B}=[2,-5]$

   b. $\vec{A}=[0,-5,1]$ and $\vec{B}=[3,3,8]$

2. Find the angle between $\vec{M}$ and $\vec{N}$ to the nearest degree given vectors $\vec{M}$ and $\vec{N}$.

   a. $\vec{M}=[5,11]$ and $\vec{N}=[4,-1]$

   b. $\vec{M}=[2,2,-1]$ and $\vec{N}=[3,5,10]$

3. Segment PQ is determined by the two points P and Q.

   Determine the angle made by $\overrightarrow{PQ}$ and the negative x-axis.

   a. $P(5,-2)$ and $Q(-3,-7)$

   b. $P(3,2,-2)$ and $Q(0,0,1)$

4. Segment MN is determined by the two points M and N.

   Determine the angle made by $\overrightarrow{MN}$ and the negative y-axis.

   a. $M(-3,-5)$ and $N(0,-4)$

   b. $M(5,5,7)$ and $N(11,9,7)$

5. The vertices of ΔPQR are $P(2,-5)$, $Q(9,-2)$, and $R(-7,1)$.

   Use $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ to find $\angle P$, use $\overrightarrow{QP}$ and $\overrightarrow{QR}$ to find $\angle Q$, and use $\overrightarrow{RP}$ and $\overrightarrow{RQ}$ to find $\angle R$.

6. The vertices of ΔDEF are $D(3,3,1)$, $E(0,-1,5)$, and $F(3,1,-1)$.

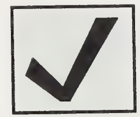
   Use $\overrightarrow{DE}$ and $\overrightarrow{DF}$ to find $\angle D$, use $\overrightarrow{EF}$ and $\overrightarrow{ED}$ to find $\angle E$, and use $\overrightarrow{FD}$ and $\overrightarrow{FE}$ to find $\angle F$.

7. Find k if $\vec{A}=[3,k,1]$ and $\vec{B}=[4,3,3k]$ are orthogonal.

8. Find k if $\vec{P}=[5,1,k]$ and $\vec{Q}=[4,k,3]$ are orthogonal.

9. Find h and k if $\vec{A}=[4,3,k]$ and $\vec{B}=[5,k,-3]$ are orthogonal to $\vec{C}=[h,2,5]$.

10. Find h and k if $\vec{P}=[8,1,-1]$ and $\vec{Q}=[0,4,3]$ are orthogonal to $\vec{R}=[2h,3,k]$.

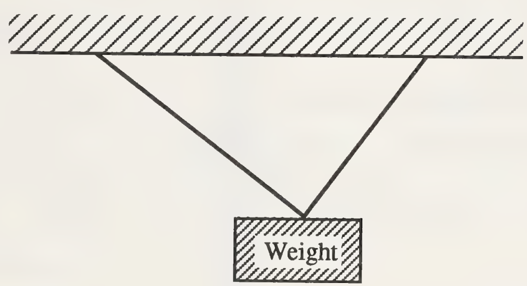


For solutions to **Activity 1**, turn to the **Appendix**, **Topic 3**.

Activity 2

Resolve a vector into two perpendicular components.

The sum of two vectors is a single vector; therefore, a single vector may be considered as the sum of two or more vectors. The process of changing a single vector to the sum of two other vectors is called the resolution of a vector. If $\vec{A}+\vec{B}=\vec{R}$, then $\vec{A}$ and $\vec{B}$ are called the components of $\vec{R}$. A vector can have an infinite number of components. Each component of a vector is the effective value of this vector in a specific direction. For example, a load is supported by two cables. To ensure that strong enough cable is used to hold the load, you must find the force of tension that will be exerted on each cable. The force of tension in each cable is the component of the weight of the load.

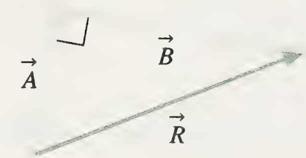


A vector may have an infinite number of components. Since this topic involves vectors in three-space, complications may arise. To avoid complications, discussion will be confined to the resolution of vectors into orthogonal components only.

It is customary and most useful to resolve a vector into two perpendicular vectors. If the two components are perpendicular to each other, then they are called **rectangular components**. It is possible to have an unlimited number of pairs of rectangular components of a vector. However, those that are collinear with the *x*- and *y*-axes are especially useful pairs.

Suppose you do not know the directions of the component vectors, but you know that they are rectangular components. Then, if $\vec{R}$ is to be resolved into any two rectangular components $\vec{A}$ and $\vec{B}$, you will have a right triangle.

According to the Pythagorean theorem, $\left|\vec{R}\right|^2=\left|\vec{A}\right|^2+\left|\vec{B}\right|^2$.



Also, according to the definition of dot product, $\vec{A}\bullet\vec{B}=0$. $\left(\vec{A}\text{ and }\vec{B}\text{ are orthogonal.}\right)$ These are relations which you can use to find the magnitudes of $\vec{A}$ and $\vec{B}$.

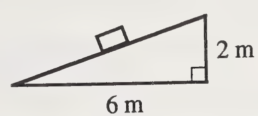
Projection of a force on another vector is the component of the force in the direction of the second vector.

Example 6

A box is pulled up a smooth inclined plane which rises 2 m in 6 m. The magnitude (in newtons) and the direction of the force is represented by the vector $[25, 30]$. Find the component of the force that pulls the box up the incline. (The force is parallel to the incline.)

Solution:

Let $\vec{F}(\text{force}) = [25, 30]$ and $\vec{S}(\text{incline}) = [6, 2]$. The magnitude of the force pulling the box up the incline is the projection of $\vec{F}$ on $\vec{S}$.



$$\begin{aligned}\text{Projection} &= \frac{[25, 30] \cdot [6, 2]}{\sqrt{6^2 + 2^2}} \\ &= \frac{(25)(6) + (30)(2)}{\sqrt{36 + 4}} \\ &\doteq \frac{210}{6.325} \\ &\doteq 33.2\end{aligned}$$

The component of the force is approximately 33.2 N.

The following example shows how the Pythagorean theorem can be used to find the components of a vector.

Example 7

Find $\left|\overrightarrow{W_1}\right|$ and $\left|\overrightarrow{W_2}\right|$ given $\vec{W} = [3, 4]$, $\overrightarrow{W_1}$ and $\overrightarrow{W_2}$ are two rectangular components of $\vec{W}$, and $\left|\overrightarrow{W_2}\right| = 2\left|\overrightarrow{W_1}\right|$.

Solution:

$$\begin{aligned}\left|\vec{W}\right|^2 &= \left|\overrightarrow{W_1}\right|^2 + \left|\overrightarrow{W_2}\right|^2 \\ &= \left|\overrightarrow{W_1}\right|^2 + \left(2\left|\overrightarrow{W_1}\right|\right)^2 \\ &= \left|\overrightarrow{W_1}\right|^2 + 4\left|\overrightarrow{W_1}\right|^2 \\ &= 5\left|\overrightarrow{W_1}\right|^2\end{aligned}$$

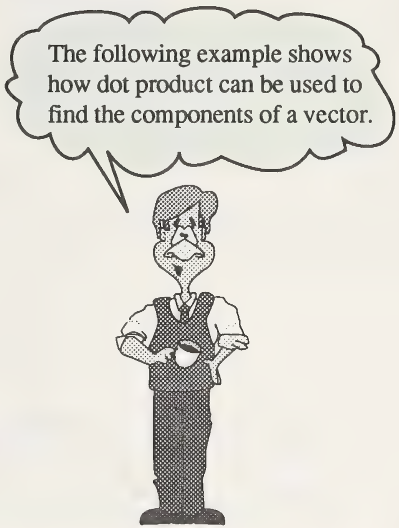
Since $\vec{W} = [3, 4]$, $\left|\vec{W}\right|^2 = 3^2 + 4^2 = 25.$

$$25 = 5\left|\overrightarrow{W_1}\right|^2$$

$$\left|\overrightarrow{W_1}\right|^2 = 5$$

$$\left|\overrightarrow{W_1}\right| = \sqrt{5}$$ (The absolute value is always positive.)

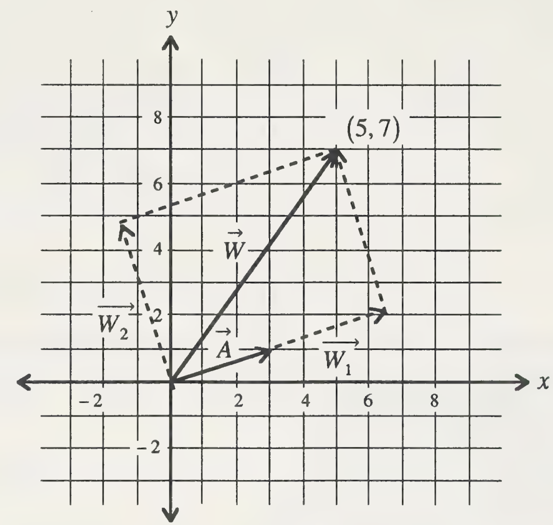
$$\left|\overrightarrow{W_2}\right| = 2\left|\overrightarrow{W_1}\right|$$
$$= 2\sqrt{5}$$



Example 8

Find $\overrightarrow{W_1}$ and $\overrightarrow{W_2}$ given $\vec{W} = [5, 7]$, $\overrightarrow{W_1}$ and $\overrightarrow{W_2}$ are two rectangular components of $\vec{W}$, and $\overrightarrow{W_1}$ is parallel to $\vec{A} = [3, 1]$.

Solution:



Since $\overrightarrow{W_1}$ and $\overrightarrow{W_2}$ are orthogonal, $\overrightarrow{W_1} \bullet \overrightarrow{W_2} = 0$.

Since $\overrightarrow{W_1}$ and $\vec{A}$ are parallel, solve for $\overrightarrow{W_1}$.

$$\begin{aligned}\overrightarrow{W_1} &= k\vec{A}\\ &= k[3,1]\\ &= [3k, k]\end{aligned}$$

Since $\overrightarrow{W_1}$ and $\overrightarrow{W_2}$ are components of $\vec{W}$, $\vec{W} = \overrightarrow{W_1} + \overrightarrow{W_2}$ (vector sum).

$$\begin{aligned}\overrightarrow{W_2} &= \vec{W} - \overrightarrow{W_1}\\ &= [5,7]-[3k,k]\\ &= [5-3k, 7-k]\end{aligned}$$

$$\begin{aligned}\overrightarrow{W_1} \bullet \overrightarrow{W_2} = [3k,k]\bullet[5-3k,7-k] &= 0\\ 0 &= 15k-9k^2+7k-k^2\\ &= -10k^2+22k\\ &= -2k(5k-11)\end{aligned}$$

$$k = 0 \text{ or } k = \frac{11}{5}$$

Therefore, $k = \frac{11}{5}$ $(k \neq 0)$.

If $k = \frac{11}{5}$, then solve for $\overrightarrow{W_1}$ and $\overrightarrow{W_2}$.

$$\begin{aligned}\overrightarrow{W_1} &= \frac{11}{5}[3,1]\\ &= \left[\frac{33}{5}, \frac{11}{5}\right]\end{aligned}$$

$$\begin{aligned}\overrightarrow{W_2} &= \left[5-3\left(\frac{11}{5}\right), 7-\frac{11}{5}\right]\\ &= \left[-\frac{8}{5}, \frac{24}{5}\right]\end{aligned}$$

To resolve a vector $\vec{W}$ into rectangular components $\overrightarrow{W_1}$ and $\overrightarrow{W_2}$, which are parallel to the *x*- and *y*-axes respectively, use the pair of noncollinear unit vectors $\vec{e}_1 = [1,0]$ and $\vec{e}_2 = [0,1]$.

If $\vec{W} = [a,b]$, then the following occurs:

$$[a,b] = a[1,0]+b[0,1]$$

$$[a,b] = a\vec{e}_1 + b\vec{e}_2$$

Look at the following example.

Example 9

Resolve vector $\vec{W} = [5, -8]$ into rectangular components if $\overrightarrow{W_1}$ and $\overrightarrow{W_2}$ are collinear with the x- and y-axes respectively.

Solution:

$$\begin{aligned}[5, -8] &= 5[1, 0] + (-8)[0, 1] \\ &= 5\vec{e}_1 + (-8)\vec{e}_2\end{aligned}$$

$$\begin{aligned}\overrightarrow{W_1} &= 5[1, 0] \\ &= [5, 0] \\ &= 5\vec{e}_1\end{aligned}$$

$$\begin{aligned}\overrightarrow{W_2} &= -8[0, 1] \\ &= [0, -8] \\ &= -8\vec{e}_2\end{aligned}$$

In three-dimensional space to resolve a vector into components collinear with each axis (that is parallel to each coordinate axis), use the unit vectors that form a basis for three-space.

$\vec{e}_1 = [1, 0, 0]$ (collinear with x-axis in three-space)

$\vec{e}_2 = [0, 1, 0]$ (collinear with y-axis in three-space)

$\vec{e}_3 = [0, 0, 1]$ (collinear with z-axis in three-space)

Example 10

Resolve vector $\vec{u} = [6, -2, 9]$ into rectangular components $\vec{u}_1, \vec{u}_2$, and $\vec{u}_3$ which are collinear with the x-, y-, and z-axes respectively.

Solution:

$$\begin{aligned}\vec{u} &= [6, -2, 9] \\ &= 6[1, 0, 0] + (-2)[0, 1, 0] + 9[0, 0, 1] \\ &= 6\vec{e}_1 + (-2)\vec{e}_2 + 9\vec{e}_3\end{aligned}$$

Therefore, $\vec{u}_1$ is $[6, 0, 0]$ or $6\vec{e}_1$,

$\vec{u}_2$ is $[0, -2, 0]$ or $-2\vec{e}_2$, and

$\vec{u}_3$ is $[0, 0, 9]$ or $9\vec{e}_3$.

In this course our discussion will be confined to the resolution of three-dimensional vectors into components parallel to the x-, y-, and z-axes.

Note: There should be no confusion using $\vec{e}_1$ and $\vec{e}_2$ as the symbols for the vectors in two-space and three-space. In each case it should be known whether you are discussing two- or three-dimensional vectors.

Now look at the following example.

Example 11

An object is moved from $P(3, -1, 5)$ to $Q(7, 3, 7)$. The magnitude of the force is 12 N. Find the magnitude of the components collinear with the x-, y-, and z-axes.

Solution:

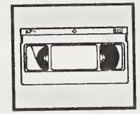
$$\overrightarrow{PQ} = [7-3, 3-(-1), 7-5]$$
$$= [4, 4, 2]$$

A force vector $\left(\vec{F}\right)$ of magnitude 12 N is in the same direction as $\overrightarrow{PQ}$. Therefore, the force vector must be $k[4, 4, 2] = [4k, 4k, 2k]$.

The magnitude of the force is as follows:

$$\left|\vec{F}\right| = \sqrt{(4k)^2 + (4k)^2 + (2k)^2}$$
$$12 = \sqrt{36k^2}$$
$$k = 2$$
$$\therefore \vec{F} = 2[4, 4, 2]$$
$$= [8, 8, 4]$$

Therefore, the component collinear with the x-axis is 8 N, the component collinear with the y-axis is 8 N, and the component collinear with the z-axis is 4 N.



For a reinforcement of the concepts in this topic, you may wish to view the video titled **Resolution of Vectors**. This video is program 16 in the *Catch 31*[1] series.

Now it is your turn.

Do the odd- or even-numbered questions.

1. Find the rectangular components collinear with the x- and y-axes (or x-, y-, and z-axes) for the following vectors.

   a. $\vec{A} = [5, 9]$

   b. $\vec{B} = [3, 7, 11]$

2. Find the rectangular components collinear with the x- and y-axes (or x-, y-, and z-axes) for the following vectors.

   a. $\vec{A} = [-2, 8]$

   b. $\vec{B} = [6, 1, 7]$

---

[1] *Catch 31* is a title of ACCESS Network.

3. Find the magnitudes of two equal rectangular components of $\vec{W} = [5, -5]$.

4. Find the magnitudes of the two equal rectangular components of $\vec{u} = [8, -7]$.

5. The two rectangular components of $\vec{u}$ are $\vec{u}_1$ and $\vec{u}_2$ where $3\left|\vec{u}_1\right| = 1\left|\vec{u}_2\right|$. Find $\left|\vec{u}_1\right|$ and $\left|\vec{u}_2\right|$ if $\vec{u} = [5, 15]$.

6. The two rectangular components of $\vec{u}$ are $\vec{u}_1$ and $\vec{u}_2$ where $\left|\vec{u}_1\right| = 5\left|\vec{u}_2\right|$. Find $\left|\vec{u}_1\right|$ and $\left|\vec{u}_2\right|$ if $\vec{u} = (4, 12)$.

7. Resolve vector $\vec{W} = [8, 10]$ into two rectangular components $\overrightarrow{W_1}$ and $\overrightarrow{W_2}$ such that $\overrightarrow{W_1}$ is parallel to vector $\vec{v} = [3, 1]$.

8. Resolve vector $\vec{R} = [-2, 6, 1]$ into two rectangular components $\overrightarrow{R_1}$ and $\overrightarrow{R_2}$ such that $\overrightarrow{R_1}$ is parallel to vector $\vec{S} = [-5, 1, 1]$.

9. An object is moved from point $P(4, 1, 7)$ to $Q(-3, 4, 2)$. The magnitude of the force is 50 N. Find the magnitude of the components with the *x*-, *y*-, and *z*-axes.

10. An object is moved from the origin to $P(5, 3, 4)$. The magnitude of the force is 40 N. Find the magnitude of the components with the *x*-, *y*-, and *z*-axes.

11. A worker is rolling a syrup drum up a smooth inclined plane that rises 3 m in 8 m. If the vector $\vec{F} = [7, 4]$ represents the magnitude and direction of the force exerted by the worker, what force parallel to the plane is used to move the drum up the plane?

12. A worker is pulling a block of wood up a smooth inclined plane that rises 2 m in 7 m. A force is exerted on the block of wood in the direction and magnitude of the vector $\overrightarrow{PQ}$ with $P(1, 4)$ and $Q(3, 8)$. What force parallel to the plane is exerted on the block of wood?

For solutions to **Activity 2**, turn to the **Appendix, Topic 3**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

You may decide to do both.

Extra Help

If two vectors are given, the angle between these two vectors is determined by the following formula:

$$\cos\ \theta = \frac{\vec{A} \bullet \vec{B}}{|\vec{A}||\vec{B}|}$$

If $\vec{A}$ and $\vec{B}$ are two vectors, then $|\vec{A}|$ and $|\vec{B}|$ are the magnitudes of the two vectors and $\vec{A} \bullet \vec{B}$ is their dot product.

Example 12

Find the angle between $\vec{A} = [5, 9]$ and $\vec{B} = [7, 6]$.

Solution:

$$\begin{aligned}\cos\ \theta &= \frac{[5,9]\bullet[7,6]}{\sqrt{5^2+9^2}\ \sqrt{(7)^2+(6)^2}}\\ &= \frac{(5)(7)+(9)(6)}{\sqrt{106}\ \sqrt{85}}\\ &= \frac{89}{\sqrt{9010}}\\ &\doteq 0.9376\\ \therefore \theta &\doteq 20.3^\circ\end{aligned}$$

If two points are given, a vector can be determined. You have to know which point is the initial point and which is the terminal point of the vector. Always subtract the initial point from the terminal point. If the coordinate axis forms a side of the angle, then use $[\pm 1, 0]$ or $[\pm 1, 0, 0]$ to represent the vector on the x-axis, $[0, \pm 1]$ or $[0, \pm 1, 0]$ to represent the vector on the y-axis, and $[0, 0, \pm 1]$ to represent the vector on the z-axis.

Example 13

Segment MN is determined by $M(5,1)$ and $N(2,7)$. Determine the angle made by $\overrightarrow{MN}$ and the negative x-axis.

Solution:

M is the initial point and N is the terminal point of segment MN.

$$\begin{aligned}\overrightarrow{MN} &= [2-5, 7-1]\\ &= [-3, 6]\end{aligned}$$

The vector on the negative x-axis is represented by $[-1, 0]$.

$$\begin{aligned}\cos\,\theta &= \frac{[-3,6]\bullet[-1,0]}{\sqrt{(-3)^2+6^2}\,\sqrt{(-1)^2+0}}\\ &= \frac{3+0}{\sqrt{9+36}\,\sqrt{1}}\\ &= \frac{3}{\sqrt{45}}\\ &\doteq 0.4472\\ \therefore \theta &\doteq 63.4^\circ\end{aligned}$$

Therefore, the angle is approximately 63.4°.

If two vectors are orthogonal, their dot product equals zero.

Example 14

Find k if $\vec{M} = [10, 5]$ and $\vec{N} = [-1, -k]$ are orthogonal.

Solution:

$$\begin{aligned}\vec{M}\bullet\vec{N} = [10,5]\bullet[-1,-k] &= 0\\ -10-5k &= 0\\ 5k &= -10\\ k &= -2\end{aligned}$$

Any vector can be resolved into two rectangular components. If $\overrightarrow{W_1}$ and $\overrightarrow{W_2}$ are the two rectangular components of $\vec{W}$, then $\overrightarrow{W_1}\bullet\overrightarrow{W_2} = 0$ and $\left|\vec{W}\right|^2 = \left|\overrightarrow{W_1}\right|^2 + \left|\overrightarrow{W_2}\right|^2$.

Example 15

Find $\left|\overrightarrow{A_1}\right|$ and $\left|\overrightarrow{A_2}\right|$ given $\vec{A}=[7,15]$, $\overrightarrow{A_1}$ and $\overrightarrow{A_2}$ are the two rectangular components of $\vec{A}$, and $\left|\overrightarrow{A_1}\right|=3\left|\overrightarrow{A_2}\right|$.

Solution:

$$\left|\vec{A}\right|=\sqrt{7^2+15^2}$$
$$=\sqrt{49+225}$$
$$=\sqrt{274}$$

$$\left|\vec{A}\right|^2=\left|\overrightarrow{A_1}\right|^2+\left|\overrightarrow{A_2}\right|^2$$

$$274=\left|\overrightarrow{A_1}\right|^2+\left|\overrightarrow{A_2}\right|^2$$

$$274=\left(3\left|\overrightarrow{A_2}\right|\right)^2+\left|\overrightarrow{A_2}\right|^2$$

$$\left|\overrightarrow{A_2}\right|^2=27.4$$

$$\left|\overrightarrow{A_2}\right|\doteq 5.234$$

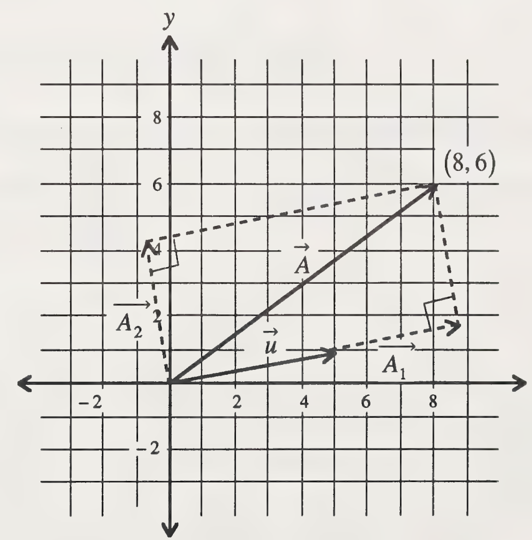
$$\therefore \left|\overrightarrow{A_1}\right|\doteq 15.7$$

The next example shows you how to find the rectangular components of a vector if the direction of one component is given.

Example 16

Find $\overrightarrow{A_1}$ and $\overrightarrow{A_2}$ given $\vec{A}=[8,6]$, $\overrightarrow{A_1}$ and $\overrightarrow{A_2}$ are two rectangular components of $\vec{A}$, and $\overrightarrow{A_1}$ is collinear with $\vec{u}=[5,1]$.

Solution:



$$\overrightarrow{A_1} = k\vec{u}$$
$$= k[5,1]$$
$$= [5k, k]$$

$$\vec{A} = \overrightarrow{A_1} + \overrightarrow{A_2}$$

$$\overrightarrow{A_2} = \vec{A} - \overrightarrow{A_1}$$
$$= [8,6] - [5k - k]$$
$$= [8 - 5k, 6 - k]$$

$$\overrightarrow{A_1} \bullet \overrightarrow{A_2} = [5k, k] \bullet [8 - 5k, 6 - k]$$
$$= 40k - 25k^2 + 6k - k^2$$
$$= 46k - 26k^2$$

$$\overrightarrow{A_1} \bullet \overrightarrow{A_2} = 0$$
$$46k - 26k^2 = 0$$
$$2k(23 - 13k) = 0$$

$$k = 0 \text{ or } k = \frac{23}{13}$$

Since $k \neq 0$, $k = \frac{23}{13}$.

$$\overrightarrow{A_1} = \frac{23}{13}[5,1]$$
$$= \left[\frac{115}{13}, \frac{23}{13}\right]$$

$$\overrightarrow{A_2} = \left[8 - \frac{115}{13}, 6 - \frac{23}{13}\right]$$
$$= \left[-\frac{11}{13}, \frac{55}{13}\right]$$

Now try the following questions.

1. Find the angle θ between the the vectors $\vec{A} = [-2, 9]$ and $\vec{B} = [-3, 1]$.

2. Segment AB is determined by $A(-5,1)$ and $B(-2,6)$. Determine the angle θ made by $\overrightarrow{AB}$ and the positive y-axis.

3. Find $\left|\overrightarrow{A_1}\right|$ and $\left|\overrightarrow{A_2}\right|$ given $\vec{A} = [4, 8]$, $\overrightarrow{A_1}$ and $\overrightarrow{A_2}$ are the two rectangular components of $\vec{A}$, and $\left|\overrightarrow{A_1}\right| = \frac{1}{2}\left|\overrightarrow{A_2}\right|$.

4. Find $\overrightarrow{A_1}$ and $\overrightarrow{A_2}$ given $\vec{A} = [-3, -7]$, $\overrightarrow{A_1}$ and $\overrightarrow{A_2}$ are the two rectangular components of $\vec{A}$, and $\overrightarrow{A_1}$ is collinear with $\vec{u} = [-3, 1]$.

For solutions to **Extra Help**, turn to the **Appendix**, **Topic 3**.

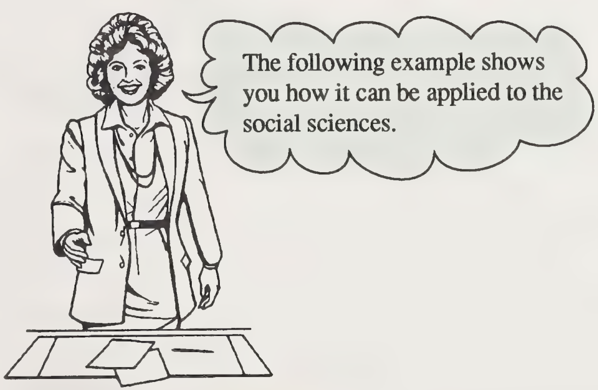
Extensions

What is a vector?

Some examples of vectors are as follows:

- a coordinate in two-space such as $[3, 5]$
- a coordinate in three-space such as $[10, 1, 4]$
- the cost of four types of candy such as $[\$1.20, \$0.75, \$2.20, \$3.00]$

An ordered set of *n* scalars is a vector of order *n*. Each scalar in an ordered set is a component of the vector. If a vector has *n* components, then it is called an *n*-dimensional vector. *n*-dimensional vectors are very useful in engineering, agriculture, and social sciences.



Example 17

There are four industries: the construction industry which builds houses and commercial buildings, the chemical industry, the refining industry which produces gasoline and heating oil, and the utility industry which supplies natural gas and electricity. There are three types of consumers: the schools, the general public, and the farmers. The schools need new buildings, repairs, gasoline for school buses, natural gas, and electricity. The schools need one unit of construction, five units of gasoline, and seven units of natural gas and electricity. The demand vector for the schools will be expressed as $\overrightarrow{D_S} = [1,0,5,7]$.

The other demand vectors are as follows:

- $\overrightarrow{D_G} = [3, 1, 4, 6]$ (the general public)
- $\overrightarrow{D_F} = [2, 5, 6, 5]$ (the farmers)
- $\overrightarrow{D_1} = [0, 1, 4, 3]$ (the construction industry)
- $\overrightarrow{D_2} = [2, 0, 7, 4]$ (the chemical industry)
- $\overrightarrow{D_3} = [1, 4, 0, 2]$ (the refining industry)
- $\overrightarrow{D_4} = [1, 0, 6, 0]$ (the utility industry)

The cost of construction is \$40 per unit, the cost of chemicals is \$15 per unit, the price of fuel is \$25 per unit, and the price of electricity is \$30 per unit.

Determine the profit or loss of the construction industry.

Solution:

The total demand on the industries is as follows:

$$\vec{D} = \overrightarrow{D_S} + \overrightarrow{D_G} + \overrightarrow{D_F} + \overrightarrow{D_1} + \overrightarrow{D_2} + \overrightarrow{D_3} + \overrightarrow{D_4}$$

$$\vec{D} = [10, 11, 32, 27]$$

Use a vector to express the costs:

$$\vec{C} = [40, 15, 25, 30]$$

Assume that the industries produce exactly what the consumers want. The inner product of the two vectors $\overrightarrow{D_1}$ and $\vec{C}$ represents the operating cost of the construction industry.

$$\begin{aligned}\overrightarrow{D_1} \cdot \vec{C} &= [0, 1, 4, 3] \cdot [40, 15, 25, 30] \\ &= 0 + 15 + 100 + 90 \\ &= \$205\end{aligned}$$

The income of the construction industry is $\$40 \times 10 = \$400$.

Therefore, the profit of the construction industry is $\$400 - \$205 = \$195$.

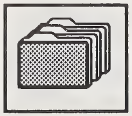
The vectors used here have more than three components. You would not be able to draw its diagram. A point in n-space contains an n number of elements. It is a row vector with n components. A row vector can be identified with a row matrix. You will learn more about matrices in **Unit 9**.

Now do the following question.

Find the profit or loss for each of the other industries in Example 17.

For solutions to **Extensions**, turn to the **Appendix**, **Topic 3**.

Unit Summary



What You Have Learned

Having completed this unit, you should be able to do the following:

- Find the inner product of two vectors using both the algebraic and geometric definition of inner product of vectors.
- Find the angle between two vectors.
- Solve problems involving perpendicular vectors.
- Find the projection of one vector on another vector, and solve problems relating to projections.
- Resolve a vector into perpendicular components.
- Find the work done by applying the inner product of vectors.

You are now ready to complete the **Unit Assignment**.

Appendix

Solutions

Review

Topic 1 Definition and Evaluation

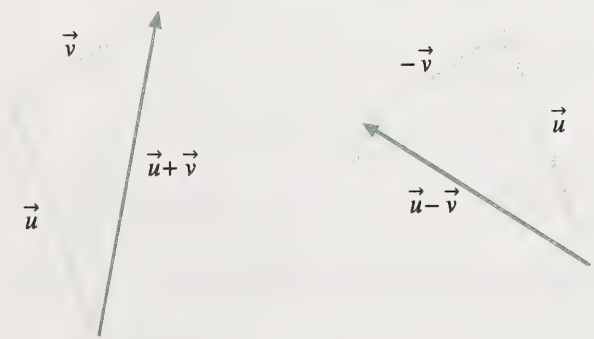
Topic 2 Projections and Work

Topic 3 Vector Angles and Resolution of a Vector into Two Perpendicular Components

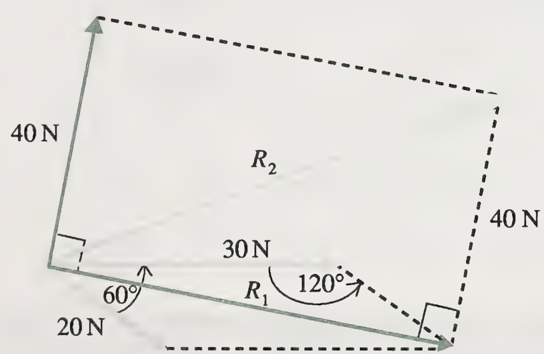
Appendix
Solutions

Review

1.



2.



$$\left|\overrightarrow{R_1}\right|^2 = (20)^2 + (30)^2 - 2(20)(30)\cos 120°$$
$$= 400 + 900 - 1200(-0.5)$$
$$= 400 + 900 + 600$$
$$= 1900$$

$$\left|\overrightarrow{R_2}\right|^2 = 1900 + 40^2$$
$$= 1900 + 1600$$
$$= 3500$$
$$\left|\overrightarrow{R_2}\right| \doteq 59$$

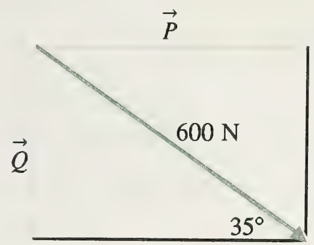
The magnitude of the resultant is approximately 59 N.

3. $\overrightarrow{PQ} = [4-1, 0-(-3), (-2)-5]$
$= [3, 3, -7]$

4. $\vec{u} = [3, -2, 1]$
$\left|\vec{u}\right| = \sqrt{3^2 + (-2)^2 + 1}$
$= \sqrt{9+4+1}$
$= \sqrt{14}$

5. $3[3,-2,1]=[9,-6,3]$

6.



$$\sin 35° = \frac{|\vec{Q}|}{600\text{ N}}$$

$$|\vec{Q}| \doteq (600\text{ N})(0.5736)$$

$$\doteq 344\text{ N}$$

$$\cos 35° = \frac{|\vec{P}|}{600\text{ N}}$$

$$|\vec{P}| \doteq (600\text{ N})(0.8192)$$

$$\doteq 491\text{ N}$$

7. $\vec{A}+\vec{B}=[3+2,1+5]$

$=[5,6]$

Exploring Topic 1

Activity 1

Define inner product in three different ways, and calculate expressions using inner product.

1. a. $\vec{A}\bullet\vec{B}=[-3,1]\bullet[5,2]$

$=(-3)(5)+(1)(2)$

$=-15+2$

$=-13$

b. $\vec{A}\bullet\vec{B}=[2,-1,5]\bullet[0,3,6]$

$=(2)(0)+(-1)(3)+(5)(6)$

$=0+(-3)+30$

$=27$

c. $\vec{A}\bullet\vec{B}=[a,\pi,1]\bullet[b,\pi,2]$

$=(a)(b)+(\pi)(\pi)+(1)(2)$

$=ab+\pi^2+2$

2. a. $\vec{F} \bullet \vec{S} = [5, 1] \bullet [-2, 3]$

$= (5)(-2) + (1)(3)$

$= -10 + 3$

$= -7$

b. $\vec{F} \bullet \vec{S} = [3, 1, 2] \bullet [-2, 0, 1]$

$= (3)(-2) + (1)(0) + (2)(1)$

$= -6 + 0 + 2$

$= -4$

c. $\vec{F} \bullet \vec{S} = [a, 2, \pi] \bullet \left[b, 3, \frac{1}{\pi}\right]$

$= ab + 6 + 1$

$= ab + 7$

3. a. $\vec{A} \bullet \vec{A} = [3, -2] \bullet [3, -2]$

$= (3)(3) + (-2)(-2)$

$= 9 + 4$

$= 13$

b. $\vec{A} \bullet \vec{A} = [0, -1, 5] \bullet [0, -1, 5]$

$= 0^2 + (-1)^2 + 5^2$

$= 0 + 1 + 25$

$= 26$

4. a. $\vec{B} \bullet \vec{B} = [-3, -2] \bullet [-3, -2]$

$= (-3)^2 + (-2)^2$

$= 9 + 4$

$= 13$

b. $\vec{B} \bullet \vec{B} = [-2, -1, 3] \bullet [-2, -1, 3]$

$= (-2)^2 + (-1)^2 + 3^2$

$= 4 + 1 + 9$

$= 14$

5. $[3, -5, 2m] \bullet [3, 2, -1] = -2$

$(3)(3) + (-5)(2) + (2m)(-1) = -2$

$9 - 10 - 2m = -2$

$-1 - 2m = -2$

$2 - 1 = 2m$

$1 = 2m$

$m = \frac{1}{2}$

6. $[n, -2, 1] \bullet [3, 1, n] = 4$

$3n - 2 + n = 4$

$4n - 2 = 4$

$4n = 6$

$n = \frac{3}{2}$

7. a. $3\vec{A}=3[3,-1]$

$=[9,-3]$

$3\vec{A}\bullet\vec{B}=[9,-3]\bullet[5,2]$

$=(9)(5)+(-3)(2)$

$=45-6$

$=39$

b. $\vec{A}-\vec{B}=[3,-1]-[5,2]$

$=(3-5,-1-2)$

$=[-2,-3]$

$\left(\vec{A}-\vec{B}\right)\bullet C=[-2,-3]\bullet[0,4]$

$=(-2)(0)+(-3)(4)$

$=-12$

8. a. $\vec{D}\bullet\vec{E}=[5,-1]\bullet[1,3]$

$=5-3$

$=2$

$\vec{F}\bullet\vec{D}=[2,2]\bullet[5,-1]$

$=10-2$

$=8$

$\therefore\vec{D}\bullet\vec{E}+\vec{F}\bullet\vec{D}=10$

b. $3\vec{D}=3[5,-1]$

$=[15,-3]$

$\vec{E}+\vec{F}=[1,3]+[2,2]$

$=[3,5]$

$3\vec{D}\bullet\left(\vec{E}+\vec{F}\right)=[15,-3]\bullet[3,5]$

$=(15)(3)+(-3)(5)$

$=30$

9. a. $3\vec{B}=3[-1,-2,2]$

$=[-3,-6,6]$

$3\vec{A}=3[3,1,2]$

$=[9,3,6]$

LS	RS
$3\vec{B}\bullet\vec{A}$	$\vec{B}\bullet 3\vec{A}$
$[-3,-6,6]\bullet[3,1,2]$	$[-1,-2,2]\bullet[9,3,6]$
$-9-6+12$	$-9-6+12$
-3	-3

LS = RS

b.

LS	RS
$\vec{A} \bullet \vec{A}$	$\left\lvert\vec{A}\right\rvert^2$
$[3,1,2]\bullet[3,1,2]$	$\left(\sqrt{3^2+1^2+2^2}\right)^2$
$(3)(3)+(1)(1)+(2)(2)$	$3^2+1^2+2^2$
$9+1+4$	$9+1+4$
14	14

LS = RS

10. a. $5\vec{D}=5[-2,1,2]$

$=[-10,5,10]$

$5\vec{C}=5[5,1,0]$

$=[25,5,0]$

LS	RS
$\vec{C} \bullet 5\vec{D}$	$5\vec{C} \bullet \vec{D}$
$[5,1,0]\bullet[-10,5,10]$	$[25,5,0]\bullet[-2,1,2]$
$(5)(-10)+(1)(5)+(0)(10)$	$(25)(-2)+(5)(1)+(0)(2)$
-45	-45

LS = RS

b.

LS	RS
$\vec{D} \bullet \vec{D}$	$\left\lvert\vec{D}\right\rvert^2$
$[-2,1,2]\bullet[-2,1,2]$	$\left(\sqrt{(-2)^2+1^2+(2)^2}\right)^2$
$4+1+4$	$(-2)^2+1^2+2^2$
9	9

LS = RS

11. a. $\vec{A}=[-3,0]$ and $\vec{B}=[0,5]$

$\vec{A}$ is on the x-axis and $\vec{B}$ is on the y-axis. Therefore, θ is 90°.

$$\vec{A} \bullet \vec{B}=[-3,0]\bullet[0,5]$$
$$=(-3)(0)+0(5)$$
$$=0$$

$$\left\lvert\vec{A}\right\rvert\left\lvert\vec{B}\right\rvert \cos\,\theta=\left\lvert\sqrt{(-3)^2+0^2}\right\rvert\left\lvert\sqrt{0^2+5^2}\right\rvert \cos\,90^\circ$$
$$=(3)(5)(0)$$
$$=0$$

Therefore, $\vec{A} \bullet \vec{B}=\left\lvert\vec{A}\right\rvert\left\lvert\vec{B}\right\rvert \cos\,\theta$.

b. $\vec{A}=[-3,6]$ and $\vec{B}=[-9,18]$

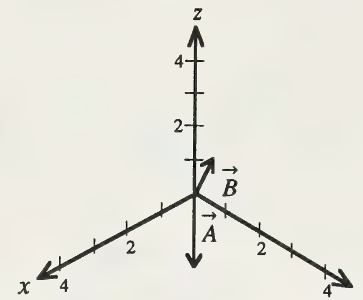
Since $\vec{B}=3\vec{A}$, $\vec{A}$ and $\vec{B}$ are collinear and $\theta=0°$.

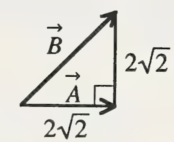
$$\begin{aligned}\vec{A}\bullet\vec{B}&=[-3,6]\bullet[-9,18]\\&=(-3)(-9)+(6)(18)\\&=27+108\\&=135\end{aligned}$$

$$\begin{aligned}|\vec{A}||\vec{B}|\cos\theta&=\left|\sqrt{(-3)^2+6^2}\right|\left|\sqrt{(-9)^2+18^2}\right|\cos 0°\\&=\left|\sqrt{9+36}\right|\left|\sqrt{81+324}\right|\cos 0°\\&=\left(\sqrt{45}\sqrt{405}\right)(1)\\&=\left(3\sqrt{5}\right)\left(9\sqrt{5}\right)\\&=135\end{aligned}$$

Therefore, $\vec{A}\bullet\vec{B}=|\vec{A}||\vec{B}|\cos\theta$.

c. $\vec{A}=[2,2,0]$ and $\vec{B}=\left[2,2,2\sqrt{2}\right]$





$\left(\text{side view of } \vec{A} \text{ and } \vec{B}\right)$

The x- and y-coordinates of $\vec{A}$ and $\vec{B}$ are equal, and the z-coordinate of $\vec{A}$ is 0; thus, $\vec{A}$ is on the xy-plane and the two vectors $\vec{A}$ and $\vec{B}$ determine a right triangle. Since the z-value of $\vec{B}=|\vec{A}|=\sqrt{2^2+2^2+0}=2\sqrt{2}$, then $\vec{A}$ and $\vec{B}$ form an isosceles triangle and the angle between $\vec{A}$ and $\vec{B}$ must be 45°.

$$\vec{A} \bullet \vec{B} = [2, 2, 0] \bullet \left[2, 2, 2\sqrt{2}\right]$$
$$= (2)(2) + (2)(2) + 0\left(2\sqrt{2}\right)$$
$$= 4 + 4 + 0$$
$$= 8$$

$$\left|\vec{A}\right|\left|\vec{B}\right| \cos 45° = \left|\sqrt{2^2 + 2^2 + 0^2}\right|\left|\sqrt{2^2 + 2^2 + \left(2\sqrt{2}\right)^2}\right| \cos 45°$$
$$= \sqrt{8}(4) \cos 45°$$
$$= \left(8\sqrt{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$
$$= 8$$

Therefore, $\vec{A} \bullet \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \cos 45°$.

12. a. $\vec{A} = [-5, 0]$ and $\vec{B} = [0, 4]$

Since $\vec{A}$ is on the x-axis and $\vec{B}$ is on the y-axis, $\theta = 90°$.

$$\vec{A} \bullet \vec{B} = [-5, 0] \bullet [0, 4]$$
$$= (-5)(0) + (0)(4)$$
$$= 0$$

$$\left|\vec{A}\right|\left|\vec{B}\right| \cos \theta = \left|\sqrt{(-5)^2 + 0}\right|\left|\sqrt{0 + 4^2}\right| \cos 90°$$
$$= (5)(4)(0)$$
$$= 0$$

Therefore, $\vec{A} \bullet \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \cos \theta$.

b. $\vec{A} = [-2, 4]$ and $\vec{B} = [-6, 12]$

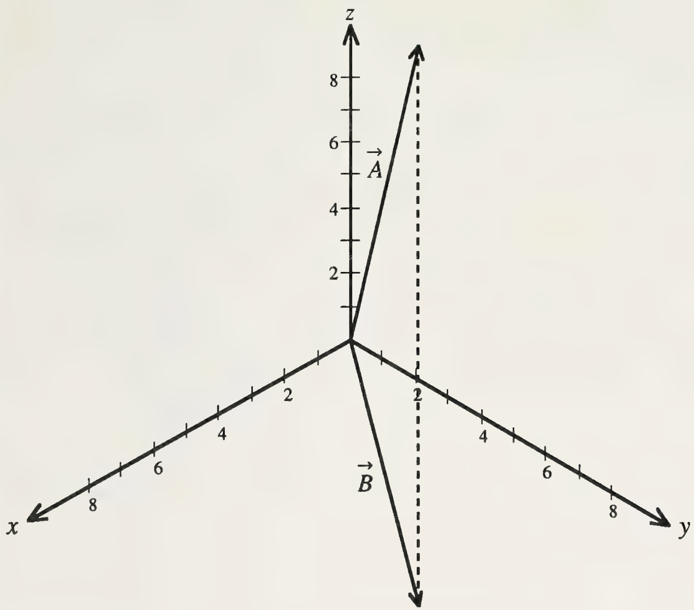
Since $\vec{B} = 3\vec{A}$, $\vec{A}$ and $\vec{B}$ are collinear and $\theta = 0°$.

$$\vec{A} \bullet \vec{B} = [-2, 4] \bullet [-6, 12]$$
$$= (-2)(-6) + (4)(12)$$
$$= 60$$

$$\left|\vec{A}\right|\left|\vec{B}\right| \cos \theta = \left|\sqrt{(-2)^2 + 4^2}\right|\left|\sqrt{(-6)^2 + (12)^2}\right| \cos 0°$$
$$= \left(\sqrt{20}\sqrt{180}\right)(1)$$
$$= \sqrt{3600}$$
$$= 60$$

Therefore, $\vec{A} \bullet \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \cos \theta$.

c. $\vec{A} = \left[6, 8, 10\sqrt{3}\right]$ and $\vec{B} = [6, 8, 0]$



The x- and y-coordinates of $\vec{A}$ and $\vec{B}$ are equal, and the z-value of $\vec{B}$ is zero; thus, $\vec{B}$ is on the xy-plane and $\vec{A}$ and $\vec{B}$ form a right triangle. Since $\left|\vec{B}\right| = 10$ and the z-value of $\vec{A}$ is $10\sqrt{3}$, the ratio is $1{:}\sqrt{3}$. The angle between $\vec{A}$ and $\vec{B}$ must be 60°.

$$\begin{aligned}\vec{A} \bullet \vec{B} &= \left[6, 8, 10\sqrt{3}\right] \bullet [6, 8, 0] \\ &= 36 + 64 + 0 \\ &= 100\end{aligned}$$

$$\begin{aligned}\left|\vec{A}\right|\left|\vec{B}\right| \cos\theta &= \left|\sqrt{6^2 + 8^2 + \left(10\sqrt{3}\right)^2}\right|\left|\sqrt{6^2 + 8^2 + 0}\right| \cos 60^\circ \\ &= \sqrt{400}\,\sqrt{100}\left(\frac{1}{2}\right) \\ &= (20)(10)\left(\frac{1}{2}\right) \\ &= 100\end{aligned}$$

Therefore, $\vec{A} \bullet \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \cos\theta$.

13. $\vec{A} = [3, 1, -2]$ and $\vec{B} = [5, 0, 7]$

$$\begin{aligned}\cos\theta = \frac{\vec{A} \bullet \vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|} &= \frac{[3, 1, -2] \bullet [5, 0, 7]}{\left|\sqrt{3^2 + 1^2 + (-2)^2}\right|\left|\sqrt{5^2 + 0 + 7^2}\right|} \\ &= \frac{(3)(5) + 0 + (-2)(7)}{\sqrt{14}\,\sqrt{74}} \\ &\doteq \frac{1}{32.187} \\ &\doteq 0.0311\end{aligned}$$

Therefore, $\theta \doteq 88.2^\circ$.

14. $\vec{u}=[-3,-5,1]$ and $\vec{v}=[0,-2,3]$

$$\cos\,\theta=\frac{\vec{u}\bullet\vec{v}}{|\vec{u}||\vec{v}|}=\frac{[-3,-5,1]\bullet[0,-2,3]}{\left|\sqrt{(-3)^2+(-5)^2+1^2}\right|\left|\sqrt{0+(-2)^2+3^2}\right|}$$
$$=\frac{0+10+3}{\sqrt{35}\,\sqrt{13}}$$
$$\doteq\frac{13}{21.33}$$
$$\doteq 0.6094$$

Therefore, $\theta\doteq 52.5°$.

Extra Help

1. $\vec{A}\bullet\vec{B}=[2,3]\bullet[1,6]$
$$=(2)(1)+(3)(6)$$
$$=2+18$$
$$=20$$

2. $\vec{A}\bullet\vec{A}=[2,3]\bullet[2,3]$
$$=2^2+3^2$$
$$=4+9$$
$$=13$$

3. $\vec{A}\bullet\vec{B}=[0,2,1]\bullet[3,2,7]$
$$=(0)(3)+(2)(2)+(1)(7)$$
$$=0+4+7$$
$$=11$$

4. Since $\vec{A}$ is on the x-axis and $\vec{B}$ is on the y-axis, $\theta=90°$.

$$\vec{A}\bullet\vec{B}=[-3,0]\bullet[0,-2]$$
$$=(-3)(0)+0(-2)$$
$$=0$$

$$|\vec{A}||\vec{B}|\cos\,\theta=\left|\sqrt{(-3)^2+0}\right|\left|\sqrt{0+(-2)^2}\right|\cos\,90°$$
$$=(3)(2)(0)$$
$$=0$$

Therefore, $\vec{A}\bullet\vec{B}=|\vec{A}||\vec{B}|\cos\,\theta$.

5. $\vec{A}\bullet\vec{B} = [5,-1]\bullet[3,2]$

$$= (5)(3)+(-1)(2)$$
$$= 13$$

$$\left|\vec{A}\right| = \left|\sqrt{5^2+(-1)^2}\right| = \sqrt{26} \qquad \left|\vec{B}\right| = \left|\sqrt{3^2+2^2}\right| = 13$$

$$\cos\ \theta = \frac{\vec{A}\bullet\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}$$
$$= \frac{13}{\sqrt{26}\sqrt{13}}$$
$$= \frac{13}{13\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}$$

Therefore, $\theta = 45°$.

Extensions

1. $\vec{A}\times\vec{B} = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]$

$$= [(5)(-1)-(-1)(-1), (-1)(-2)-(3)(-1), (3)(-1)-(5)(-2)]$$
$$= [-6,5,7]$$

2. $\theta = 180° - 80°$

$$= 100°$$

$$\left|\vec{S}\right| = 60 \text{ cm}$$
$$= 0.6 \text{ m}$$

$$\left|\vec{F}\right| = 30 \text{ N}$$

$$\left|\vec{F}\times\vec{S}\right| = (0.6)(30)\sin 100°$$
$$\doteq 18(0.9848)$$
$$\doteq 17.7$$

The magnitude of the amount is approximately 17.7 N•m.

3. $\vec{A}\bullet\vec{B} = [(5)(9)-(-2)(0), (-2)(-3)-(3)(9), (3)(0)-(5)(-3)]$

$$= [45,-21,15]$$

$$\left|\vec{A}\bullet\vec{B}\right| = \sqrt{45^2+(-21)^2+(15)^2}$$
$$= \sqrt{2025+441+225}$$
$$= \sqrt{2691}$$
$$\doteq 51.87$$

The area is approximately 51.87 units^2.

Exploring Topic 2

Activity 1

Define scalar projection of a vector and solve related problems.

1. a. Scalar projection of $\vec{A}$ on $\vec{B} = \dfrac{[3,1]\bullet[0,-2]}{\sqrt{0^2+(-2)^2}}$

$$= \frac{0-2}{\sqrt{4}}$$

$$= -1$$

b. Scalar projection of $\vec{A}$ on $\vec{B} = \dfrac{[2,-1,3]\bullet[5,0,-4]}{\sqrt{5^2+0+(-4)^2}}$

$$= \frac{10+0-12}{\sqrt{41}}$$

$$= \frac{-2}{\sqrt{41}}$$

2. a. Scalar projection of $\vec{A}$ on $\vec{B} = \dfrac{[-2,-3]\bullet[-5,8]}{\sqrt{(-5)^2+8^2}}$

$$= \frac{10-24}{\sqrt{89}}$$

$$= -\frac{14}{\sqrt{89}}$$

b. Scalar projection of $\vec{A}$ on $\vec{B} = \dfrac{[5,6,-2]\bullet[3,0,7]}{\sqrt{3^2+0+7^2}}$

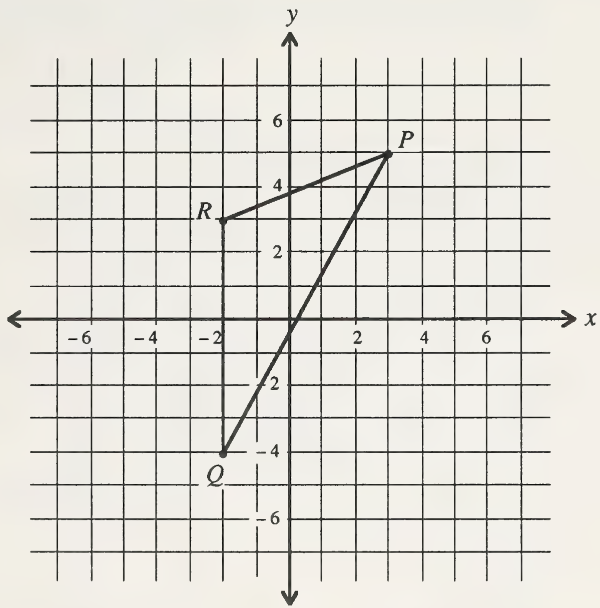
$$= \frac{15+0-14}{\sqrt{58}}$$

$$= \frac{1}{\sqrt{58}}$$

3. $\overrightarrow{PQ} = [-2-3, -4-5]$
$= [-5, -9]$

$\overrightarrow{PR} = [-2-3, 3-5]$
$= [-5, -2]$

$\overrightarrow{RQ} = [-2+2, -4-3]$
$= [0, -7]$



$$\text{Scalar projection of } \overrightarrow{PR} \text{ on } \overrightarrow{PQ} = \frac{[-5,-2]\bullet[-5,-9]}{\sqrt{(-5)^2+(-9)^2}}$$
$$= \frac{25+18}{\sqrt{106}}$$
$$= \frac{43}{\sqrt{106}}$$

$$\text{Scalar projection of } \overrightarrow{RQ} \text{ on } \overrightarrow{PQ} = \frac{[0,-7]\bullet[-5,-9]}{\sqrt{(-5)^2+(-9)^2}}$$
$$= \frac{63}{\sqrt{106}}$$

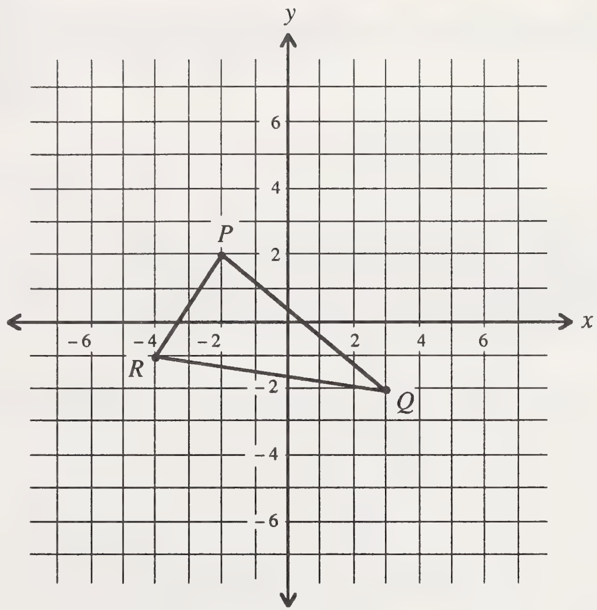
$$\text{Length of } \overrightarrow{PQ} = \left|\sqrt{(-5)^2+(-9)^2}\right| = \sqrt{106}$$

Therefore, the scalar projection of $\overrightarrow{PR}$ on $\overrightarrow{PQ}$ added to the scalar projection of $\overrightarrow{RQ}$ on $\overrightarrow{PQ}$ is equal to the length of $\overrightarrow{PQ}$.

4. $\overrightarrow{PQ} = [3+2, -2-2]$
$= [5, -4]$

$\overrightarrow{PR} = [-4+2, -1-2]$
$= [-2, -3]$

$\overrightarrow{QR} = [-4-3, -1+2]$
$= [-7, 1]$



Scalar projection of $\overrightarrow{PQ}$ on $\overrightarrow{PR} = \dfrac{[5,-4]\cdot[-2,-3]}{\sqrt{(-2)^2+(-3)^2}}$

$$= \frac{-10+12}{\sqrt{13}}$$

$$= \frac{2}{\sqrt{13}}$$

Scalar projection of $\overrightarrow{QR}$ on $\overrightarrow{PR} = \dfrac{[-7,1]\cdot[-2,-3]}{\sqrt{(-2)^2+(-3)^2}}$

$$= \frac{14-3}{\sqrt{13}}$$

$$= \frac{11}{\sqrt{13}}$$

Length of $\overrightarrow{PR} = \left|\sqrt{(-2)^2+(-3)^2}\right|$

$$= \sqrt{13}$$

The projection of $\overrightarrow{PQ}$ on $\overrightarrow{PR}$ added to the projection of $\overrightarrow{QR}$ on $\overrightarrow{PR}$ is equal to the length of $\overrightarrow{PR}$.

5. The projection of $\vec{A}$ on $\vec{B}$ is zero when $\vec{A}$ and $\vec{B}$ are perpendicular.

6. The projection of $\vec{A}$ on $\vec{B}$ equals the projection of $\vec{B}$ on $\vec{A}$ when $|\vec{A}| = |\vec{B}|$.

Activity 2

Define work and solve related problems.

1. $\vec{F} \cdot \vec{S} = [4, -3] \cdot [2, 7]$

   $= 8 - 21$

   The work done is -13 J.

2. $\vec{F} \cdot \vec{S} = [-2, 5] \cdot [-3, 4]$

   $= 6 + 20$

   The work done is 26 J.

3. $\overrightarrow{PQ} = (9-5, 0-3, 8-1)$

   $= [4, -3, 7]$

   $\overrightarrow{MN} = [0-2, 3-1, 1-4]$

   $= [-2, 2, -3]$

   $$\left|\vec{F}\right| = k\left|\overrightarrow{MN}\right|$$

   $$30 = k\left|\sqrt{(-2)^2 + (2)^2 + (-3)^2}\right|$$

   $$30 = k\left(\sqrt{4+4+9}\right)$$

   $$30 = k\left(\sqrt{17}\right)$$

   $$k = \frac{30}{\sqrt{17}}$$

   $$\vec{F} = \frac{30}{\sqrt{17}}[-2, 2, -3]$$

   $$= \left[-\frac{60}{\sqrt{17}}, \frac{60}{\sqrt{17}}, -\frac{90}{\sqrt{17}}\right]$$

   $$\vec{F} \cdot \overrightarrow{PQ} = \left[-\frac{60}{\sqrt{17}}, \frac{60}{\sqrt{17}}, -\frac{90}{\sqrt{17}}\right] \cdot [4, -3, 7]$$

   $$= \frac{-240}{\sqrt{17}} - \frac{180}{\sqrt{17}} - \frac{630}{\sqrt{17}}$$

   $$= \frac{-1050}{\sqrt{17}}$$

   $$\doteq -254.66$$

   The work done is approximately 254.66 J.

4. $\overrightarrow{PQ} = [3-2, 5+2, 7-1]$

   $= [1, 7, 6]$

   $\theta = 45°$

   $$\left|\overrightarrow{PQ}\right|(40)\cos 45° \doteq \left(\sqrt{1^2 + 7^2 + 6^2}\right)(40)(0.7071)$$

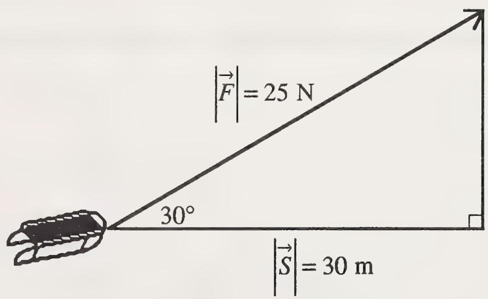
   $$\doteq \sqrt{86}\,(40)(0.7071)$$

   $$\doteq (9.2736)(40)(0.7071)$$

   $$\doteq 262.31$$

   The work done is about 262.3 J.

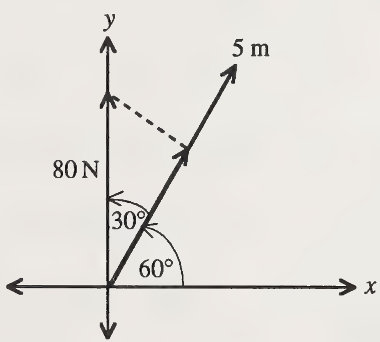
5.



$$\begin{aligned}|\vec{F}||\vec{S}|\cos 30° &\doteq (25)(30)(0.8660)\\ &\doteq 649.5\end{aligned}$$

The work done is approximately 649.5 J.

6.



The weight of the child is a downward force of 80 N. The child must exert a force slightly greater than 80 N to move vertically upward or a force greater than the component of 80 N in the direction of the staircase in order to climb the staircase.

Therefore, the work done is slightly greater than $\vec{F} \bullet \vec{D}$.

$$\begin{aligned}\vec{F} \bullet \vec{D} &= |\vec{F}||\vec{D}|\cos\theta\\ &= (80)(5)\cos 30°\\ &\doteq 400(0.8660)\\ &\doteq 346.4\end{aligned}$$

The work done is approximately 346.4 J.

Extra Help

1. Scalar projection of $\vec{B}$ on $\vec{A} = \dfrac{[-2,-5]\bullet[7,8]}{\sqrt{7^2+8^2}}$

$$= \frac{-14-40}{\sqrt{113}}$$

$$= \frac{-54}{\sqrt{113}}$$

2. $\overrightarrow{PQ} = [3,4]$

$\overrightarrow{PR} = [8,4]$

$\overrightarrow{QR} = (8-3, 4-4)$

$= [5,0]$

$$\text{Projection of } \overrightarrow{PQ} \text{ on } \overrightarrow{PR} = \frac{[3,4]\cdot[8,4]}{\sqrt{8^2+4^2}}$$
$$= \frac{24+16}{\sqrt{80}}$$
$$= \frac{40}{\sqrt{80}}$$
$$= \frac{10}{\sqrt{5}}$$

$$\text{Projection of } \overrightarrow{QR} \text{ on } \overrightarrow{PR} = \frac{[5,0]\cdot[8,4]}{\sqrt{8^2+4^2}}$$
$$= \frac{40}{\sqrt{80}}$$
$$= \frac{10}{\sqrt{5}}$$

Therefore, the projection of $\overrightarrow{PQ}$ on $\overrightarrow{PR}$ is equal to the projection of $\overrightarrow{QR}$ on $\overrightarrow{PR}$.

3. $[-3,5]\cdot[-2,9] = 6+45$
$$= 51$$
The work done is 51 J.

4. $\overrightarrow{AB} = [-2-3, 6-3]$
$$= [-5, 3]$$
$$\left|\overrightarrow{AB}\right| = \sqrt{(-5)^2+3^2}$$
$$= \sqrt{34}$$

$$\left(\sqrt{34}\right)(25)\cos 70° \doteq \sqrt{34}\,(25)(0.3420)$$
$$\doteq 49.9$$
The work done is approximately 49.9 J.

Extensions

1. Vector projection of $\vec{B}$ on $\vec{A} = \dfrac{[1,8,3]\cdot[5,0,-3]}{\left|\sqrt{5^2+0+(-3)^2}\right|^2}[5,0,-3]$
$$= \frac{5+0-9}{34}[5,0,-3]$$
$$= -\frac{2}{17}[5,0,-3]$$
$$= \left[-\frac{10}{17}, 0, \frac{6}{17}\right]$$

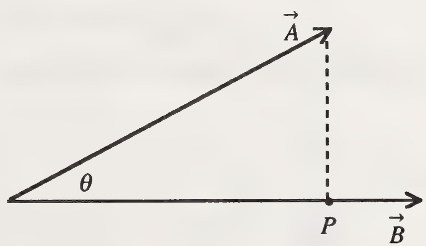
2. Vector projection of $\vec{A}$ on the y-axis $= \dfrac{[-3,5,1]\bullet[0,1,0]}{\left|\sqrt{0+1^2+0}\right|^2}[0,1,0]$

$$= \frac{5}{1}[0,1,0]$$

$$= [0,5,0]$$

Note that the vector $[0,1,0]$ or any vector with x- and z-coordinates of zero and a positive y-coordinate represents a vector along the positive y-axis.

3.



Projection of $-3\vec{A}$ on $2\vec{B} = \dfrac{\left(-3\vec{A}\right)\bullet\left(2\vec{B}\right)}{\left|2\vec{B}\right|^2}\left(2\vec{B}\right)$

$$= \frac{-3\left(\vec{A}\bullet\vec{B}\right)}{\left|\vec{B}\right|^2}\left(\vec{B}\right)$$

Since $\overrightarrow{OP} = \frac{\vec{A}\bullet\vec{B}}{\left|\vec{B}\right|^2}\left(\vec{B}\right)$ (definition of vector projection), the projection of $-3\vec{A}$ on $2\vec{B}$ is $-3\overrightarrow{OP}$.

Exploring Topic 3

Activity 1

Determine vector angles.

1. a. $\cos\theta = \dfrac{\vec{A}\bullet\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}$

$$= \frac{[3,-9]\bullet[2,-5]}{\sqrt{3^2+(-9)^2}\,\sqrt{2^2+(-5)^2}}$$

$$= \frac{6+45}{\sqrt{90}\,\sqrt{29}}$$

$$= \frac{51}{\sqrt{2610}}$$

$$\doteq 0.9983$$

$$\therefore \theta \doteq 3^\circ$$

b. $$\begin{aligned}\cos\theta &= \frac{[0,-5,1]\bullet[3,3,8]}{\sqrt{(-5)^2+1^2}\sqrt{3^2+3^2+8^2}}\\ &= \frac{0-15+8}{\sqrt{26}\sqrt{82}}\\ &\doteq \frac{-7}{46.174}\\ &\doteq -0.1516\\ \therefore \theta &\doteq 99^\circ\end{aligned}$$

2. a. $$\begin{aligned}\cos\theta &= \frac{[5,11]\bullet[4,-1]}{\sqrt{5^2+11^2}\sqrt{4^2+(-1)^2}}\\ &= \frac{20-11}{\sqrt{146}\sqrt{17}}\\ &\doteq \frac{9}{49.82}\\ &\doteq 0.1807\\ \therefore \theta &\doteq 80^\circ\end{aligned}$$

b. $$\begin{aligned}\cos\theta &= \frac{[2,2,-1]\bullet[3,5,10]}{\sqrt{2^2+2^2+(-1)^2}\sqrt{3^2+5^2+10^2}}\\ &= \frac{6+10-10}{\sqrt{9}\sqrt{134}}\\ &= \frac{2}{\sqrt{134}}\\ &\doteq 0.1728\\ \therefore \theta &\doteq 80^\circ\end{aligned}$$

3. a. $$\begin{aligned}\overrightarrow{PQ} &= [-3-5,-7+2]\\ &= [-8,-5]\end{aligned}$$

Use $[-1,0]$ to represent the negative x-axis.

$$\begin{aligned}\cos\theta &= \frac{[-8,-5]\bullet[-1,0]}{\sqrt{(-8)^2+(-5)^2}\sqrt{1^2}}\\ &= \frac{8+0}{\sqrt{89}}\\ &\doteq 0.8480\\ \therefore \theta &\doteq 32^\circ\end{aligned}$$

b. $\overrightarrow{PQ} = [-3, -2, 3]$

Use $[-1, 0, 0]$ to represent the negative x-axis.

$$\begin{aligned}\cos\,\theta &= \frac{[-3,-2,3]\bullet[-1,0,0]}{\sqrt{(-3)^2+(-2)^2+3^2}\,\sqrt{(-1)^2}}\\ &= \frac{3}{\sqrt{22}}\\ &\doteq 0.6396\\ \therefore \theta &\doteq 50^\circ\end{aligned}$$

4. a. $\overrightarrow{MN} = [0-(-3), -4-(-5)]$

$= [3, 1]$

Use $[0, -1]$ to represent the negative y-axis.

$$\begin{aligned}\cos\,\theta &= \frac{[3,1]\bullet[0,-1]}{\sqrt{3^2+1^2}\,\sqrt{(-1)^2}}\\ &= \frac{-1}{\sqrt{10}}\\ &= -0.3162\\ \therefore \theta &\doteq 108^\circ\end{aligned}$$

b. $\overrightarrow{MN} = [11-5, 9-5, 7-7]$

$= [6, 4, 0]$

Use $[0, -1, 0]$ to represent the negative y-axis.

$$\begin{aligned}\cos\,\theta &= \frac{[6,4,0]\bullet[0,-1,0]}{\sqrt{6^2+4^2}\,\sqrt{(-1)^2}}\\ &= \frac{0-4+0}{\sqrt{52}\,\sqrt{1}}\\ &\doteq \frac{-4}{7.2111}\\ &\doteq -0.5547\\ \therefore \theta &\doteq 124^\circ\end{aligned}$$

5. $\overrightarrow{PQ} = [9-2, -2+5]$

$= [7, 3]$

$\overrightarrow{PR} = [-7-2, 1+5]$

$= [-9, 6]$

$\overrightarrow{QP} = [2-9, -5+2]$

$= [-7, -3]$

$\overrightarrow{QR} = [-7-9, 1+2]$

$= [-16, 3]$

$$\overrightarrow{RP} = [2+7, -5-1] = [9, -6]$$

$$\overrightarrow{RQ} = [9+7, -2-1] = [16, -3]$$

$$\cos \angle P = \frac{[7,3]\bullet[-9,6]}{\sqrt{7^2+3^2}\sqrt{(-9)^2+6^2}} = \frac{-63+18}{\sqrt{58}\sqrt{117}} = -\frac{45}{82.377} = -0.5463$$

$$\therefore \angle P \doteq 123°$$

$$\cos \angle Q = \frac{[-7,-3]\bullet[-16,3]}{\sqrt{(-7)^2+(-3)^2}\sqrt{(-16)^2+3^2}} = \frac{112-9}{\sqrt{58}\sqrt{265}} = \frac{103}{123.976} = 0.8308$$

$$\therefore \angle Q \doteq 34°$$

$$\cos \angle R = \frac{[9,-6]\bullet[16,-3]}{\sqrt{9^2+(-6)^2}\sqrt{(16)^2+(-3)^2}} = \frac{144+18}{\sqrt{117}\sqrt{265}} = \frac{162}{176.08} = 0.9200$$

$$\therefore \angle R \doteq 23°$$

6. $\overrightarrow{DE} = [0-3, -1-3, 5-1] = [-3, -4, 4]$

$\overrightarrow{DF} = [3-3, 1-3, -1-1] = [0, -2, -2]$

$\overrightarrow{EF} = [3-0, 1+1, -1-5] = [3, 2, -6]$

$\overrightarrow{ED} = [3-0, 3+1, 1-5] = [3, 4, -4]$

$\overrightarrow{FD} = [3-3, 3-1, 1+1] = [0, 2, 2]$

$\overrightarrow{FE} = [0-3, -1-1, 5+1] = [-3, -2, 6]$

$$\cos \angle D = \frac{[-3,-4,4]\bullet[0,-2,-2]}{\sqrt{(-3)^2+(-4)^2+4^2}\sqrt{(-2)^2+(-2)^2}} = \frac{0+8-8}{\sqrt{41}\sqrt{8}} = 0$$

$$\therefore \angle D = 90°$$

$$\cos \angle E = \frac{[3,2,-6]\cdot[3,4,-4]}{\sqrt{3^2+2^2+(-6)^2}\sqrt{3^2+4^2+(-4)^2}}$$

$$= \frac{9+8+24}{\sqrt{49}\sqrt{41}}$$

$$= \frac{41}{\sqrt{2009}}$$

$$\doteq \frac{41}{44.82}$$

$$\doteq 0.9147$$

$$\therefore \angle E \doteq 24°$$

$$\cos \angle F = \frac{[0,2,2]\cdot[-3,-2,6]}{\sqrt{2^2+2^2}\sqrt{(-3)^2+(-2)^2+6^2}}$$

$$= \frac{-4+12}{\sqrt{8}\sqrt{49}}$$

$$\doteq \frac{8}{19.799}$$

$$\doteq 0.4041$$

$$\therefore \angle F \doteq 66°$$

7. $\vec{A}\cdot\vec{B}=0$

$$[3,k,1]\cdot[4,3,3k]=0$$
$$12+3k+3k=0$$
$$-12=6k$$
$$k=-2$$

8. $\vec{P}\cdot\vec{Q}=0$

$$[5,1,k]\cdot[4,k,3]=0$$
$$20+k+3k=0$$
$$20+4k=0$$
$$4k=-20$$
$$k=-5$$

9. $\vec{A}\cdot\vec{C}=0$

$$[4,3,k]\cdot[h,2,5]=0$$
$$4h+6+5k=0 \quad (1)$$

$\vec{B}\cdot\vec{C}=0$

$$[5,k,-3]\cdot[h,2,5]=0$$
$$5h+2k-15=0 \quad (2)$$

$$5\times(2): 25h+10k-75=0 \quad (3)$$
$$2\times(1): \underline{8h+10k+12=0} \quad (4)$$
$$(3)-(4): 17h-87=0$$
$$17h=87$$
$$h=\frac{87}{17}$$

Substitute $h=\frac{87}{17}$ in 2.

$$5\left(\frac{87}{17}\right)+2k-15=0$$
$$\frac{435}{17}+2k-15=0$$
$$2k=15-\frac{435}{17}$$
$$2k=-\frac{180}{17}$$
$$k=-\frac{90}{17}$$

10. $\vec{P}\bullet\vec{R}=0$

$$[8,1,-1]\bullet[2h,3,k]=0$$
$$16h+3-k=0 \quad (1)$$

$\vec{Q}\bullet\vec{R}=0$

$$[0,4,3]\bullet[2h,3,k]=0$$
$$12+3k=0$$
$$3k=-12$$
$$k=-4$$

Substitute $k=-4$ in 1.

$$16h+3+4=0$$
$$16h=-7$$
$$h=-\frac{7}{16}$$

Activity 2

Resolve a vector into two perpendicular components.

1. a. $\vec{A}=[5,9]$

   The two components are $[5,0]$ and $[0,9]$.

   b. $\vec{B}=[3,7,11]$

   The three components are $[3,0,0]$, $[0,7,0]$, and $[0,0,11]$.

2. a. $\vec{A}=[-2,8]$

   The two components are $[-2,0]$ and $[0,8]$.

   b. $\vec{B}=[6,1,7]$

   The three components are $[6,0,0]$, $[0,1,0]$, and $[0,0,7]$.

3. Let $\overrightarrow{W_1}$ and $\overrightarrow{W_2}$ be the two components.

$$\left|\overrightarrow{W_1}\right|=\left|\overrightarrow{W_2}\right|$$
$$\left|\vec{W}\right|^2=\left|\overrightarrow{W_1}\right|^2+\left|\overrightarrow{W_2}\right|^2$$

$$\left|\sqrt{5^2+(-5)^2}\right|^2 = 2\left|\overrightarrow{W_1}\right|^2$$

$$50 = 2\left|\overrightarrow{W_1}\right|^2$$

$$\left|\overrightarrow{W_1}\right|^2 = 25$$

$$\therefore \left|\overrightarrow{W_1}\right| = \left|\overrightarrow{W_2}\right| = 5$$

4. Let $\vec{u}_1$ and $\vec{u}_2$ be the two components.

$$\left|\vec{u}_1\right| = \left|\vec{u}_2\right|$$

$$\left|\vec{u}\right|^2 = \left|\vec{u}_1\right|^2 + \left|\vec{u}_2\right|^2$$

$$\left|\sqrt{8^2+(-7)^2}\right|^2 = 2\left|\vec{u}_1\right|^2$$

$$113 = 2\left|\vec{u}_1\right|^2$$

$$\left|\vec{u}_1\right|^2 = 56.5$$

$$\therefore \left|\vec{u}_1\right| = \left|\vec{u}_2\right| \doteq 7.52$$

5.

$$3\left|\vec{u}_1\right| = \left|\vec{u}_2\right|$$

$$\left|\vec{u}\right|^2 = \left|\vec{u}_1\right|^2 + \left|\vec{u}_2\right|^2$$

$$\left|\sqrt{5^2+15^2}\right|^2 = \left|\vec{u}_1\right|^2 + \left(3\left|\vec{u}_1\right|\right)^2$$

$$250 = \left|\vec{u}_1\right|^2 + 9\left|\vec{u}_1\right|^2$$

$$= 10\left|\vec{u}_1\right|^2$$

$$\left|\vec{u}_1\right|^2 = 25$$

$$\left|\vec{u}_1\right| = 5$$

$$\left|\vec{u}_2\right| = 3\left|\vec{u}_1\right|$$

$$= (3)(5)$$

$$= 15$$

6. $$\left|\vec{u}_1\right| = 5\left|\vec{u}_2\right|$$

$$\left|\vec{u}\right|^2 = \left|\vec{u}_1\right|^2 + \left|\vec{u}_2\right|^2$$

$$\left|\sqrt{4^2+12^2}\right|^2 = \left(5\left|\vec{u}_2\right|\right)^2 + \left|\vec{u}_2\right|^2$$

$$160 = 26\left|\vec{u}_2\right|^2$$

$$\left|\vec{u}_2\right|^2 \doteq 6.1538$$

$$\left|\vec{u}_2\right| \doteq 2.48$$

$$\therefore \left|\vec{u}_1\right| \doteq (5)(2.48)$$

$$\doteq 12.4$$

7. $\vec{W} = [8,10]$, $\vec{v} = [3,1]$

$$\overrightarrow{W_1} = k[3,1]$$

$$= [3k, k]$$

$$\vec{W} = \overrightarrow{W_1} + \overrightarrow{W_2}$$

$$[8,10] = [3k,k] + \overrightarrow{W_2}$$

$$\overrightarrow{W_2} = [8,10] - [3k,k]$$

$$= [8-3k, 10-k]$$

$$\overrightarrow{W_1} \cdot \overrightarrow{W_2} = 0$$

$$[3k,k] \cdot [8-3k, 10-k] = 0$$

$$24k - 9k^2 + 10k - k^2 = 0$$

$$-10k^2 + 34k = 0$$

$$-2k(5k-17) = 0$$

$k = 0$ or $k = \frac{17}{5}$

Since $k \neq 0$, $k = \frac{17}{5}$.

$$\overrightarrow{W_1} = \frac{17}{5}[3,1]$$

$$= \left[\frac{51}{5}, \frac{17}{5}\right]$$

$$\overrightarrow{W_2} = \left[8 - (3)\frac{17}{5}, 10 - \frac{17}{5}\right]$$

$$= \left[-\frac{11}{5}, \frac{33}{5}\right]$$

8. $\overrightarrow{R_1} = k[-5,1,1]$

$= [-5k, k, k]$

$\vec{R} = \overrightarrow{R_1} + \overrightarrow{R_2}$

$$\begin{aligned}\overrightarrow{R_2} &= \vec{R} - \overrightarrow{R_1}\\ &= [-2,6,1]-[-5k,k,k]\\ &= [-2+5k, 6-k, 1-k]\end{aligned}$$

$$\begin{aligned}\overrightarrow{R_1}\bullet\overrightarrow{R_2} &= 0\\ [-5k,k,k]\bullet[-2+5k,6-k,1-k] &= 0\\ 10k-25k^2+6k-k^2+k-k^2 &= 0\\ 17k-27k^2 &= 0\\ k(17-27k) &= 0\end{aligned}$$

$k = 0$ or $k = \dfrac{17}{27}$

Since $k \neq 0$, $k = \frac{17}{27}$.

$$\begin{aligned}\overrightarrow{R_1} &= \frac{17}{27}[-5,1,1]\\ &= \left[\frac{-85}{27}, \frac{17}{27}, \frac{17}{27}\right]\end{aligned}$$

$$\begin{aligned}\overrightarrow{R_2} &= \left[-2+(5)\frac{17}{27}, 6-\frac{17}{27}, 1-\frac{17}{27}\right]\\ &= \left[\frac{31}{27}, \frac{145}{27}, \frac{10}{27}\right]\end{aligned}$$

9. $\overrightarrow{PQ} = [-3-4, 4-1, 2-7]$

$= [-7, 3, -5]$

$$\begin{aligned}\vec{F} &= k[-7,3,-5]\\ &= [-7k, 3k, -5k]\end{aligned}$$

$$\begin{aligned}\left|\vec{F}\right| = 50 &= \sqrt{(-7k)^2+(3k)^2+(-5k)^2}\\ 50 &= \sqrt{49k^2+9k^2+25k^2}\\ 50 &= \sqrt{83k^2}\\ k &= \frac{50}{\sqrt{83}}\end{aligned}$$

$$\begin{aligned}\vec{F} &= \frac{50}{\sqrt{83}}[-7,3,-5]\\ &= \left[\frac{-350}{\sqrt{83}}, \frac{150}{\sqrt{83}}, \frac{-250}{\sqrt{83}}\right]\end{aligned}$$

Therefore, the *x*-component is $\left[\frac{-350}{\sqrt{83}}, 0, 0\right]$, the *y*-component is $\left[0, \frac{150}{\sqrt{83}}, 0\right]$, and the *z*-component is $\left[0, 0, \frac{-250}{\sqrt{83}}\right]$.

10. $\overrightarrow{OP} = [5, 3, 4]$

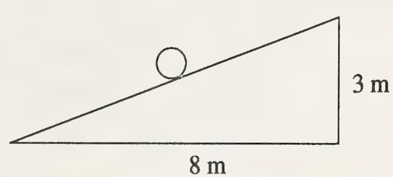
$$\begin{aligned}\vec{F} &= k[5, 3, 4] \\ &= [5k, 3k, 4k]\end{aligned}$$

$$\begin{aligned}|\vec{F}| = 40 &= \sqrt{(5k)^2 + (3k)^2 + (4k)^2} \\ 40 &= \sqrt{50k^2} \\ k &= \frac{40}{\sqrt{50}} \\ &= \frac{40}{5\sqrt{2}} \\ &= \frac{8}{\sqrt{2}} \\ &= 4\sqrt{2}\end{aligned}$$

$\therefore \vec{F} = [20\sqrt{2}, 12\sqrt{2}, 16\sqrt{2}]$

The x-component is $[20\sqrt{2}, 0, 0]$, the y-component is $[0, 12\sqrt{2}, 0]$, and the z-component is $[0, 0, 16\sqrt{2}]$.

11.



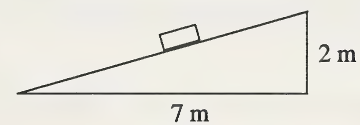
$\vec{F} = [7, 4]$

$\vec{S} = [8, 3]$

$$\begin{aligned}\text{Projection of } \vec{F} \text{ on } \vec{S} &= \frac{[7, 4] \cdot [8, 3]}{\sqrt{8^2 + 3^2}} \\ &= \frac{56 + 12}{\sqrt{73}} \\ &= \frac{68}{\sqrt{73}} \\ &\doteq 7.96\end{aligned}$$

The force used is approximately 7.96 N.

12.



$$\begin{aligned}\vec{F} = \overrightarrow{PQ} &= [3 - 1, 8 - 4] \\ &= [2, 4]\end{aligned}$$

$\vec{S} = [7, 2]$

$$\begin{aligned}\text{Projection of } \vec{F} \text{ on } \vec{S} &= \frac{[2, 4] \cdot [7, 2]}{\sqrt{7^2 + 2^2}} \\ &= \frac{14 + 8}{\sqrt{53}} \\ &\doteq \frac{22}{7.28} \\ &\doteq 3.02\end{aligned}$$

The force exerted is approximately 3.02 N.

1. $$\cos\ \theta = \frac{[-2,9]\bullet[-3,1]}{\sqrt{(-2)^2+9^2}\ \sqrt{(-3)^2+1^2}}$$
$$= \frac{6+9}{\sqrt{85}\ \sqrt{10}}$$
$$\doteq \frac{15}{29.155}$$
$$\doteq 0.5145$$
$$\therefore \theta \doteq 59°$$

2. $$\overrightarrow{AB} = [-2+5, 6-1]$$
$$= [3,5]$$

Use $[0,1]$ to represent the positive y-axis.

$$= \frac{[3,5]\bullet[0,1]}{\sqrt{3^2+5^2}\ \sqrt{1^2}}$$
$$= \frac{5}{\sqrt{34}}$$
$$\doteq 0.8575$$
$$\therefore \theta \doteq 31°$$

3. $$|(4,8)|^2 = \left(\frac{1}{2}\left|\overrightarrow{A_2}\right|\right)^2 + \left|\overrightarrow{A_2}\right|^2$$
$$\left|\sqrt{4^2+8^2}\right|^2 = \frac{1}{4}\left|\overrightarrow{A_2}\right|^2 + \left|\overrightarrow{A_2}\right|^2$$
$$80 = \frac{5}{4}\left|\overrightarrow{A_2}\right|^2$$
$$\left|\overrightarrow{A_2}\right|^2 = (80)\frac{4}{5}$$
$$= 64$$
$$\therefore \left|\overrightarrow{A_2}\right| = 8$$
$$\left|\overrightarrow{A_1}\right| = \frac{1}{2}(8)$$
$$= 4$$

4. $$\overrightarrow{A_1} = k[-3,1]$$
$$= [-3k, k]$$
$$\vec{A} = \overrightarrow{A_1} + \overrightarrow{A_2}$$
$$[-3,-7] = [-3k,k] + \overrightarrow{A_2}$$
$$\overrightarrow{A_2} = [-3,-7] - [-3k,k]$$
$$= [-3+3k, -7-k]$$

$$\overrightarrow{A_1} \cdot \overrightarrow{A_2} = 0$$

$$\begin{aligned}[-3k, k] \cdot [-3+3k, -7-k] &= 0\\ 9k - 9k^2 - 7k - k^2 &= 0\\ 2k - 10k^2 &= 0\\ 2k(1-5k) &= 0\end{aligned}$$

$k = 0$ or $k = \frac{1}{5}$

Since $k \neq 0$, $k = \frac{1}{5}$.

$$\begin{aligned}\overrightarrow{A_1} &= \frac{1}{5}[-3, 1]\\ &= \left[-\frac{3}{5}, \frac{1}{5}\right]\end{aligned}$$

$$\begin{aligned}\overrightarrow{A_2} &= \left[-3+\frac{3}{5}, -7-\frac{1}{5}\right]\\ &= \left[-\frac{12}{5}, -\frac{36}{5}\right]\end{aligned}$$

Extensions

The chemical industry calculations are as follows:

$$\begin{aligned}\overrightarrow{D_2} \cdot \vec{C} &= [2, 0, 7, 4] \cdot [40, 15, 25, 30]\\ &= \$80 + \$0 + \$175 + \$120\\ &= \$375\end{aligned}$$

The income is $\$15 \times 11 = \$165$.
Thus, the loss of the chemical industry is $\$375 - \$165 = \$210$.

The refining industry calculations are as follows:

$$\begin{aligned}\overrightarrow{D_3} \cdot \vec{C} &= [1, 4, 0, 2] \cdot [40, 15, 25, 30]\\ &= \$40 + \$60 + \$0 + \$60\\ &= \$160\end{aligned}$$

The income is $32 \times \$25 = \$800$.
Thus, the profit of the refining industry is $\$800 - \$160 = \$640$.

The utility industry calculations are as follows:

$$\begin{aligned}\overrightarrow{D_4} \cdot \vec{C} &= [1, 0, 6, 0] \cdot [40, 15, 25, 30]\\ &= \$40 + \$0 + \$150 + \$0\\ &= \$190\end{aligned}$$

The income is $27 \times \$30 = \$810$.
Thus, the profit of the utility industry is $\$810 - \$190 = \$620$.

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